Carbon footprint & linear algebra:
a short introduction to input-output analysis

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How does one compute the carbon emissions associated with the purchase of a product? When using a carbon footprint calculator such as WWF's, some questions pertain specifically to our purchases in different sectors of economy, giving us some ranges of values to choose from. In this short piece, we propose to have a look at one of the methods used to crunch those numbers.

1 Introduction

We are interested in calculating upstream emissions associated with a purchase, i.e all the carbon emissions that occurred through production. The term upstream is used in opposition to downstream emissions, ie those resulting from the use of a product or service and not its purchase.

Let's say you would like to compute the upstream carbon emissions coming from the purchase of a newspaper. You might start by drawing a tree to take into account the different parts that come into producing it: the paper and the ink printed on it. You can then go one step further, dividing the paper into the wood pulp and the water, etc... Then summing up the carbon emissions associated with all the nodes could give you an estimation.

You will however quickly encounter two problems: some interdependencies creating cycles (rather annoying for a tree) and knowing when to stop (there might be an infinite number of layers to take into account).

The environmentally extended input-output analysis remedies to these two problems, by taking into account all of the intermediate sales that occur before a final product is sold to a customer using basic linear algebra.

In order to do an environmentally extended input output analysis (EEIO), the following pieces of data are needed:

- a balanced input-output table (balanced meaning that the total inputs and total ouputs are equal)

- a measure of direct environmental impacts associated with each sector (in our case, carbon emissions)

2 What's an input-output table?

An input output table is a simple double-entry table used to represent exchanges between economic sectors within a country, or describe import-export relations between several nations. You can find such tables for Switzerland at https://www.bfs.admin.ch/bfs/en/home/statistics/national-economy/input-output.html.

In this short article, we will work on a toy model directly taken from Justin Kitzes' article. Let us assume that your economic system has two sectors, agriculture and manufacture, denoted Ag and Ma respectively. We represent in a table their inputs and outputs as follows:

	Ag	Ma	FD	TO
Ag	8	5	3	16
Ma	4	2	6	12
VA	4	5		
TI	16	12		

The first row can be read "The agriculture sector sold for 8 units of goods to agriculture, for 5 units to manufacture, and for 3 units of final products to consumers, for a total output of $8 + 5 + 5 = 16$ units".
We define below all abbreviations appearing.

Definition 2.1. *The **final demand**, denoted FD, is the total money spent by final consumers to buy finished products.*
*The **total output**, denoted TO, is the total amount sold to all sectors and consumers, while TI denotes the **total input**. Note that TO=TI.*
*The **value added** is the economic benefit realized by the sector.*

From now on, we will assume the unit to be the dollar.

3 How do carbon emissions factor in?

We now need a quantitative measure of environmental impact for the economic sectors considered: in our case, carbon emissions.
Let us therefore assume that the Ag sector emits 8 tons of carbon annually, and the Ma sector 4 tons. Using this and the input-output table above, we can now give a few definitions.

Definition 3.1. *The **direct intensity vector** f is given by the direct emissions for each sector divided by the total output for each sector. It measures the emissions associated to 1\$ of output for each sector.*

In our example, the direct intensity vector is given by $f = \begin{pmatrix} 8/16 & 4/12 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.33 \end{pmatrix}$. It is in tons/dollar.

Remark. Now, if we simply multiply this direct intensity vector by the final vector demand $\begin{pmatrix} 3 & 6 \end{pmatrix}$, it yields $3 \times 0.5 + 0.33 \times 6 = 1.5 + 2 = 3.5$ tons of carbon emissions. This certainly doesn't account for all 12 tons of carbon emissions produced by both sectors, most of which are "hidden" in the intermediate products sold between the Ag and Ma sectors.

Definition 3.2. *The **technical coefficients matrix** A is given by the amount of input one sector must receive from every sector in order to produce 1\$ of output.*

In our example, $A = \begin{pmatrix} 8/16 & 5/12 \\ 4/16 & 2/12 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.42 \\ 0.25 & 0.17 \end{pmatrix}$.
So in order to produce 1\$ of output, the Ma sector requires 0.42\$ worth of goods and services from

the Ag sector and 0.17\$ of input from Ma itself.

This further layer should be taken into account when computing carbon emissions: when using the direct intensity vector, it yields an additional $0.42 \times 0.5 + 0.17 \times 0.33 = 0.26$ tons of carbon emissions. We can now repeat the process through an infinite number of layers: of course, to produce 0.42\$ from the Ag sector, it requires 0.42×0.5\$ from Ag and 0.42×0.25\$ from Ma, etc...

Definition 3.3. *The **total intensity vector** F gives the total carbon emissions associated with 1\$ of output of final demand.*

Computing the total intensity vector is then a simple addition of the iterations of the process described above:

$$F = f \cdot I + f \cdot A + f \cdot AA + \cdots = f \cdot (I + A + A^2 + A^3 + \dots)$$

This leads to the following result:

Lemma 1. *The total intensity vector F is given by $F := f(I - A)^{-1}$.*

The computation of the total intensity vector of our example yields:

$$F = \begin{pmatrix} 0.5 & 0.33 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.42 \\ 0.25 & 0.17 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 1.6 & 1.2 \end{pmatrix}$$

4   What do we do with this?

Well, first, now we can compute the carbon emissions associated to the purchase of a final product from the agrioculture or manufacture sector: it is simply the price of the item multiplied by the corresponding coordinate.

This total intensity vector can also be used to compute the *consumption-based* emissions associated with a sector, compared to the *production-based* emissions that we started with (8 tons for Ag, 4 tons for Ma). If we multiply the final demand vector and the total intensity vector, we of course retrieve the total of 12 tons of carbon emissions. However, the carbon emissions are somewhat reallocated: it yields 4.8 tons for the Ag sector and 7.2 for the Ma sector. Indeed, the agriculture sector may produce more carbon emissions, but since a large part of its intermediate products are used by the manufacture sector, it is now reflected in the consumption-based emissions numbers.

5   Conclusion

The use of a the input-output method applied to environmental impact is mathematically meaningful. The main difficulty in its application is of course access to the needed data: a recent input-output table and a measure of environmental impact for each sector appearing in the table. The former is quite available, see for Switzerland https://www.bfs.admin.ch/bfs/en/home/statistics/national-economy/input-output.html. The latter is harder to obtain, especially as it needs to be computed/collected in an homogeneous manner over the different sectors in order to make sense.

References

Kitzes J., An Introduction to Environmentally-Extended Input-Output Analysis, Resources 2013, 2, 489-503

Leontief W., Input-Output Economics, Oxford University Press, Oxford, 1966