Strong positivity for the skein algebras of the 4-punctured sphere and of the 1-punctured torus

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ETH-ITS Zurich

7th SwissMAP General Meeting
September 7, 2020

Topics:

- Low-dimensional topology.
- Enumerative algebraic geometry.
- String theory realizations of supersymmetric gauge theories.

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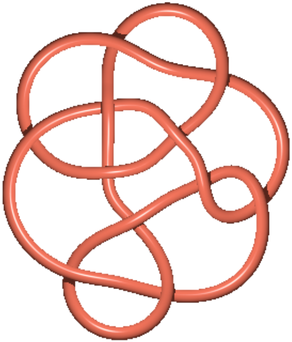
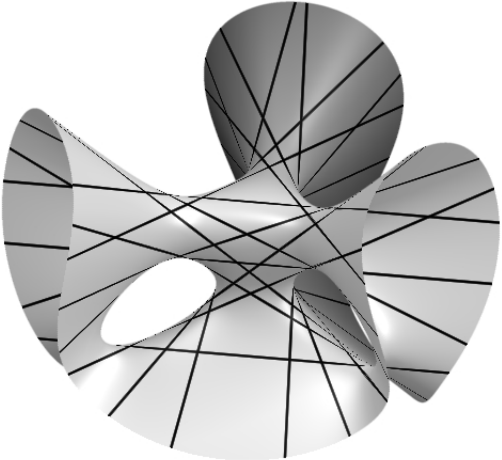
- Low-dimensional topology: knots, links...



- Enumerative algebraic geometry: 27 lines on a cubic surface (Cayley, Salmon, 1849)



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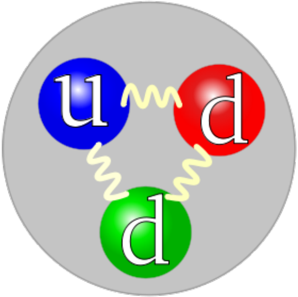

Introduction

- Result in low-dimensional topological: positive bases for Kauffman bracket skein algebras of the 4-punctured torus and the 1-punctured torus.
- Proof based on the enumerative geometry of holomorphic curves in complex cubic surfaces.
- Proof motivated by the existence of dual realizations in string/M-theory of the $\mathcal{N} = 2$ $N_f = 4$ $SU(2)$ gauge theory.

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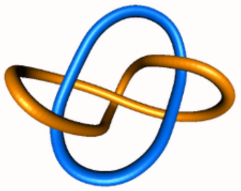
Knots, links and framing



- Knot in a manifold: a connected compact embedded 1-dimensional submanifold.
- Link in a manifold: the disjoint union of finitely many knots.
- Framing of a link: a choice of nowhere vanishing section of its normal bundle.

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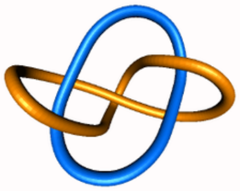

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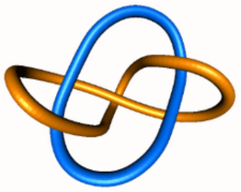

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- The Kauffman bracket skein module (Przytycki, Turaev, 1988) of an oriented 3-manifold $\mathbb{M}$ is the $\mathbb{Z}[A^{\pm}]$-module generated by isotopy classes of framed links in $\mathbb{M}$ satisfying the skein relations

- The diagrams in each relation indicate framed links that can be isotoped to identical embeddings except within the neighborhood shown, where the framing is vertical.
- The skein module of $\mathbb{M} = \mathbb{R}^3$ is $\mathbb{Z}[A^{\pm}]$ (generated by the empty link). The class of a framed link $L \subset \mathbb{R}^3$ in $\mathbb{Z}[A^{\pm}]$ is the Kauffman bracket polynomial of L (equivalent to the Jones polynomial).

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- Given an oriented 2-manifold $\mathbb{S}$, one can define a natural algebra structure on the Kauffmann bracket skein module of the 3-manifold $\mathbb{M} := \mathbb{S} \times (-1,1)$: given two framed links L_1 and L_2 in $\mathbb{S} \times (-1,1)$, and viewing the interval $(-1,1)$ as a vertical direction, the product $L_1 L_2$ is defined by placing L_1 on top of L_2.
- We denote by $\mathrm{Sk}_A(\mathbb{S})$ the resulting associative $\mathbb{Z}[A^{\pm}]$-algebra with unit. The skein algebra $\mathrm{Sk}_A(\mathbb{S})$ is in general non-commutative.

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- We consider the case where $\mathbb{S}$ is the complement $\mathbb{S}_{g,\ell}$ of a finite number ℓ of points in a compact oriented 2-manifold of genus g.
- A multicurve on $\mathbb{S}_{g,\ell}$ is the union of finitely many disjoint compact connected embedded 1-dimensional submanifolds of $\mathbb{S}_{g,\ell}$ such that none of them bounds a disc in $\mathbb{S}_{g,\ell}$. Identifying $\mathbb{S}_{g,\ell}$ with $\mathbb{S}_{g,\ell} \times \{0\} \subset \mathbb{S}_{g,\ell} \times (-1,1)$, a multicurve on $\mathbb{S}_{g,\ell}$ endowed with the vertical framing naturally defined a framed link in $\mathbb{S}_{g,\ell} \times (-1,1)$.

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Isotopy classes of multicurves form a basis of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\mathbb{Z}[A^{\pm}]$-module.

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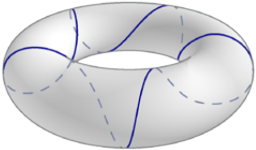

- On the closed torus $\mathbb{S}_{1,0}$, isotopy classes of multicurves are in bijection with

$$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \text{ if } n = 0\} .$$

- For every $p = (m,n) \in B(\mathbb{Z})$, denote by γ_p the corresponding isotopy class of multicurves.
- γ_p has $\gcd(m,n)$ connected components.
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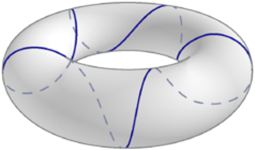

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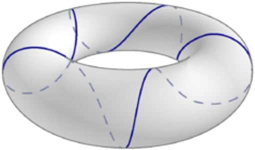

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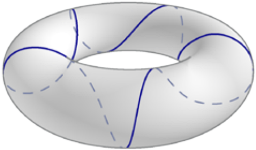

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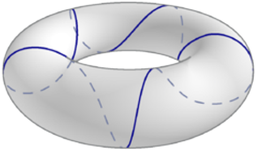

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Structure constants and positive bases

- The structure constants $C^l_{j,k} \in R$ of a basis $\{e_j\}_{j\in J}$ of an algebra $\mathcal{A}$ over a ring R are defined by

$$e_j e_k = \sum_{l\in J} C^l_{j,k} e_l \,.$$

- For the skein algebra, $R = \mathbb{Z}[A^{\pm}]$.

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A basis $\{e_j\}_{j\in J}$ of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{g,\ell}$ is called *positive* if its structure constants belong to $\mathbb{Z}_{\geq 0}[A^{\pm}]$, i.e. are Laurent polynomials in A with positive coefficients.

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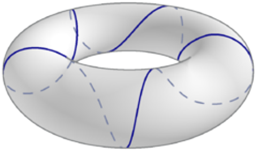

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- Two crossings to resolve:

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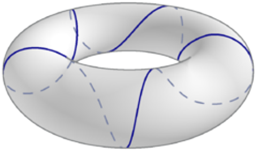

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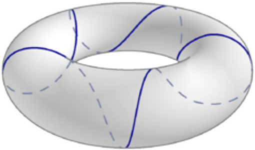

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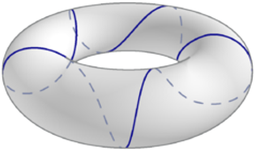

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The bracelets basis

- Let $T_n(x)$ be the Chebyshev polynomials defined by

$$T_0(x) = 1\,,\, T_1(x) = x\,,\, T_2(x) = x^2 - 2\,,$$

and for every $n \geq 2$,

$$T_{n+1}(x) = xT_n(x) - T_{n-1}(x)\,.$$

Writing $x = \lambda + \lambda^{-1}$, we have $T_n(x) = \lambda^n + \lambda^{-n}$ for every $n \geq 1$.

- Given an isotopy class γ of multicurve on $\mathbb{S}_{g,\ell}$, one can uniquely write γ in $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$ where $\gamma_1, \cdots, \gamma_r$ are all distinct isotopy classes of connected multicurves and $n_j \in \mathbb{Z}_{>0}$, and we define

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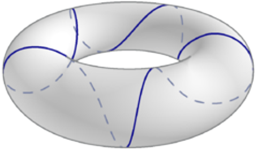

- As $T_1(x) = x$ and $T_2(x) = x^2 - 2$,

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- Positive!

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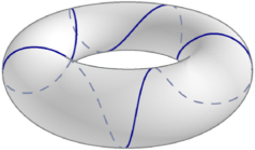

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Conjectural positivity of the bracelets basis

Conjecture (Dylan Thurston, 2013)

For every g and ℓ, the bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ is positive.

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For every g and ℓ, after setting $A=1$, the structure constants of the bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{g,\ell})$ are non-negative.

Theorem (Frohman, Gelca, 2000)

The bracelets basis $\{\mathbf{T}(\gamma)\}_\gamma$ of $\mathrm{Sk}_A(\mathbb{S}_{1,0})$ of the closed torus $\mathbb{S}_{1,0}$ is positive. In fact, for every $p_1, p_2 \in B(\mathbb{Z})$,

$$\mathbf{T}(\gamma_{p_1})\mathbf{T}(\gamma_{p_2}) = A^{\det(p_1,p_2)}\mathbf{T}(\gamma_{p_1+p_2}) + A^{-\det(p_1,p_2)}\mathbf{T}(\gamma_{p_1-p_2})\,.$$

The positivity of the bracelets basis is obvious for $\mathbb{S}_{0,0}$, $\mathbb{S}_{0,1}$, $\mathbb{S}_{0,2}$, $\mathbb{S}_{0,3}$.

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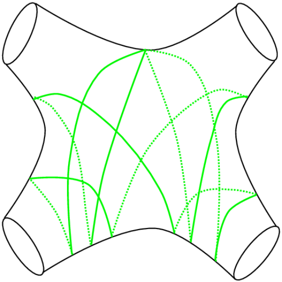
Main result

Theorem (B, 2020)

The bracelets bases $\{\mathbf{T}(\gamma)\}_\gamma$ of the skein algebras $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ and $\mathrm{Sk}_A(\mathbb{S}_{1,1})$ of the 4-punctured sphere and the 1-punctured torus are positive.

Unlike the case of the closd torus $\mathbb{S}_{1,0}$, there does not seem to exist a simple closed formula for the structure constants of the bracelets basis of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ and $\mathrm{Sk}_A(\mathbb{S}_{1,1})$.



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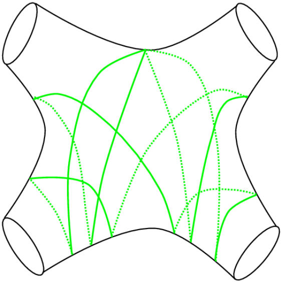
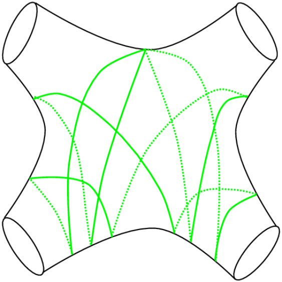


4-punctured sphere

- Focus on the case of the 4-punctured sphere $\mathbb{S}_{0,4}$.
- Peripheral curves a_1, a_2, a_3, a_4, in the center of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$, so we can view $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as a R-module, where $R = \mathbb{Z}[A^{\pm}][a_1, a_2, a_3, a_4]$.
- Isotopy classes of multicurves in $\mathbb{S}_{0,4}$ without peripheral connected components are in bjection with

$$B(\mathbb{Z}) := \mathbb{Z}^2/\langle \pm id \rangle \simeq \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid m \geq 0 \text{ if } n = 0\}.$$

(View $\mathbb{S}_{0,4}$ as a $\mathbb{Z}/2\mathbb{Z}$-quotient of a 4-punctured sphere)
- $\{\gamma_p\}_{p \in B(\mathbb{Z})}$ is a basis of $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as R-module.



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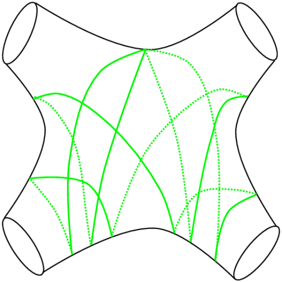

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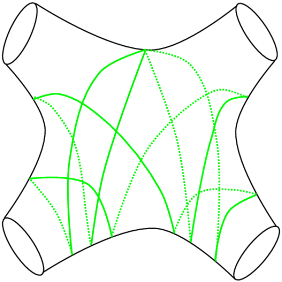

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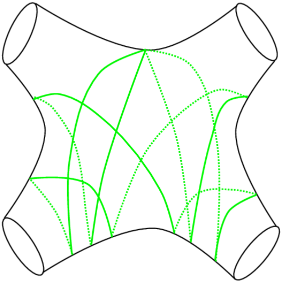

Structure of the proof

- Algorithm computing the structure constants $C^p_{p_1,p_2} \in R$ defined by

$$\mathbf{T}(\gamma_{p_1})\mathbf{T}(\gamma_{p_2}) = \sum_{p \in B(\mathbb{Z})} C^p_{p_1,p_2} \mathbf{T}(\gamma_p)\,.$$

and making manifest their positivity properties.

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"Scattering diagram": attach a power series $f_{m,n}$ to every ray in B with rational slope of primitive direction $(m,n) \in B(\mathbb{Z})$.

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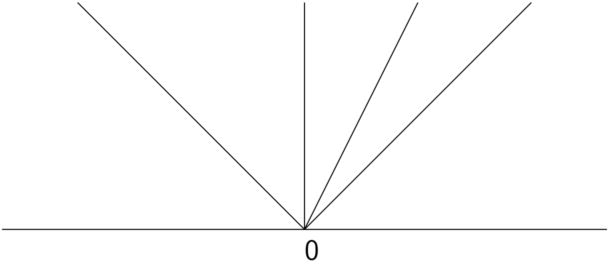

Broken line

- γ broken line in $\mathfrak{D}$ of asympotic direction $p \in B(\mathbb{Z})$ and endpoint Q
- Continuous piecewise integral affine line, bending along rays of rational slopes, decorated by monomials.
- Monomial attached to the linearity domain L of the form $c_L z^{p_L}$, where $c_L \in R$, and $-p_L$ parallel to the direction of L.
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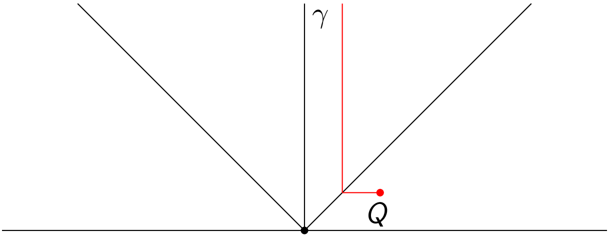

Broken line

- Bending formula for a broken line passing between the domains of linearity L and L' by bending along $\mathbb{R}_{\geq 0}(m,n)$.
- Write $m_L = c_L z^{p_L}$, $m_{L'} = c_{L'} z^{p_{L'}}$, $N = |\det((m,n), p_L)|$, and $f_{m,n} = \sum_{k\geq 0} c_k z^{-k(m,n)}$, then there exists a sequence $n = (n_k)_{k\geq 0}$ of non-negative integers with $\sum_{k\geq 0} n_k = N$ such that, denoting by

$$\beta_n \left(\prod_{k\geq 0} c_k^{n_k} \right) z^{-(\sum_{k\geq 0} n_k k)(m,n)}$$

the term proportional to $\left(\prod_{k\geq 0} c_k^{n_k}\right) z^{-(\sum_{k\geq 0} n_k k)(m,n)}$ in

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$$c_{L'} = \left(\beta_n \prod_{k\geq 0} c_k^{n_k} \right) c_L \text{ and } p_{L'} = p_L - \left(\sum_{k\geq 0} n_k k \right) (m,n) .$$

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- For every $p_1, p_2, p \in B(\mathbb{Z})$ and $Q \in B$ generic close to p, define
$$C^{\mathfrak{D},p}_{p_1,p_2}(Q) := \sum_{(\gamma_1,\gamma_2)} c(\gamma_1)c(\gamma_2)A^{2\det(s(\gamma_1),s(\gamma_2))} \in R\,,$$
where the sum is over pairs (γ_1, γ_2) of quantum broken lines for $\mathfrak{D}$ with charges p_1,p_2 and common endpoint Q, such that writing $c(\gamma_1)z^{s(\gamma_1)}$ and $c(\gamma_2)z^{s(\gamma_2)}$ the final monomials, we have $s(\gamma_1) + s(\gamma_2) = p$.
- A scattering diagram $\mathfrak{D}$ is *consistent* if for every $p_1, p_2, p \in B(\mathbb{Z})$, $C^{\mathfrak{D},p}_{p_1,p_2}(Q)$ does not depend on the choice of the point Q, and the product on the free R-module
$$\mathcal{A}_{\mathfrak{D}} := \bigoplus_{p\in B(\mathbb{Z})} R\,\vartheta_p$$
defined by
$$\vartheta_{p_1}\vartheta_{p_2} = \sum_{p\in B(\mathbb{Z})} C^{\mathfrak{D},p}_{p_1,p_2}\vartheta_p$$
is associative.

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Strategy: construct a consistent scattering diagram $\mathfrak{D}$ and an isomorphism

$$\varphi \colon \mathcal{A}_{\mathfrak{D}} \to \mathrm{Sk}_A(\mathbb{S}_{0,4})$$

such that

$$\varphi(\vartheta_p) = \mathbf{T}(\gamma_p)$$

for every $p \in B(\mathbb{Z})$.

Scattering diagram

- Notations:

$$F(r,s,y,x) := 1 + \frac{rx(1+x^2)}{(1-A^{-4}x^2)(1-A^4x^2)} + \frac{yx^2}{(1-A^{-4}x^2)(1-A^4x^2)} + \frac{sx^3(1+sx+x^2)}{(1-A^{-4}x^2)(1-x^2)^2(1-A^4x^2)}\,.$$

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- Define a scattering diagram $\mathfrak{D}$ by

$$\text{if } (m,n) = (1,0) \mod 2\,,\ f_{m,n} := F(R_{1,0}, R_{0,1}R_{1,1}, y, z^{-(m,n)})\,,$$
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Scattering diagram

- Notations:

$$F(r,s,y,x) := 1 + \frac{rx(1+x^2)}{(1-A^{-4}x^2)(1-A^4x^2)} + \frac{yx^2}{(1-A^{-4}x^2)(1-A^4x^2)}$$
$$+ \frac{sx^3(1+sx+x^2)}{(1-A^{-4}x^2)(1-x^2)^2(1-A^4x^2)}\,.$$

- $$R_{1,0} := a_1a_2 + a_3a_4\,,\quad R_{0,1} := a_1a_3 + a_2a_4\,,\quad R_{1,1} := a_1a_4 + a_2a_3\,,$$
$$y := a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + (A^2 - A^{-2})^2\,.$$

- Define a scattering diagram $\mathfrak{D}$ by

$$\text{if}\,(m,n) = (1,0) \mod 2\,,\, f_{m,n} := F(R_{1,0}, R_{0,1}R_{1,1}, y, z^{-(m,n)})\,,$$
$$\text{if}\,(m,n) = (0,1) \mod 2\,,\, f_{m,n} := F(R_{0,1}, R_{1,0}R_{1,1}, y, z^{-(m,n)})\,,$$
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Results

Theorem 1 (B, 2020)

The scattering diagram $\mathfrak{D}$ is consistent.

Theorem 2 (B, 2020)

There exists an isomorphism

$$\varphi\colon \mathcal{A}_{\mathfrak{D}} \to \mathrm{Sk}_A(\mathbb{S}_{0,4})$$

such that

$$\varphi(\vartheta_p) = \mathbf{T}(\gamma_p)$$

for every $p \in B(\mathbb{Z})$.

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Theorem 2 is proved by computations. The proof of Theorem 1 relies on enumerative algebraic geometry.

- Y: smooth projective surface over $\mathbb{C}$, D normal crossings anticanonical divisor.
- $N_{g,\beta}$: "counts" of genus g holomorphic curves in Y of class β intersecting D in a single point.
- Can use the numbers $N_{g,\beta}$ to cook up a consistent scattering diagram $\mathfrak{D}_{(Y,D)}$.
- Y: smooth cubic surface, D a triangle of lines.

Theorem 2 (B, 2020)

$$\mathfrak{D}_{(Y,D)} \simeq \mathfrak{D}\,.$$

Simplest case: $27 - 3 = 8 \times 3$ lines in Y intersecting D in a single point.

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Gauge theories from string/M-theory

- $\mathcal{T}$: $\mathcal{N} = 2$ $N_f = 4$ $SU(2)$ gauge theory.
- Realization of $\mathcal{T}$ as a class S theory: $\mathcal{N} = (2,0)$ 6d SCFT of class A_1 compactified on $\mathbb{S}_{0,4}$. Physical realization of the skein algebra $\mathrm{Sk}_A(\mathbb{S}_{0,4})$ as an algebra of supersymmetric line operators.
- U: complement of a triangle of lines D in Y, hyperkäher manifold, D_4 elliptic fibration in rotated complex structure, Σ: elliptic fiber.
- Realization of $\mathcal{T}$ from M-theory on $\mathbb{R}^{1,3} \times U \times \mathbb{R}^3$ with a $M5$-brane on $\mathbb{R}^{1,3} \times \Sigma$. Physical realization of holomorphic curves in (Y, D) as $M2$-branes determining the BPS spectrum of $\mathcal{T}$.

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Thank you for your attention!