How Long are the Arms in Dielectric Breakdown?

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Processes and Models

Many physical processes, such as mineral deposition, electrodeposition, lightnings, surface discharges, and treeing in polymers, are examples of stochastic phenomena with growth driven by harmonic measure. They tend to produce highly irregular, branching and porous clusters.

Famous *Diffusion limited aggregation* (DLA) [6] and *dielectric-breakdown model* (DBM) [5] were introduced in the 1980's to model behavior of such processes, and have been a great challenge for mathematicians and physicists ever since.

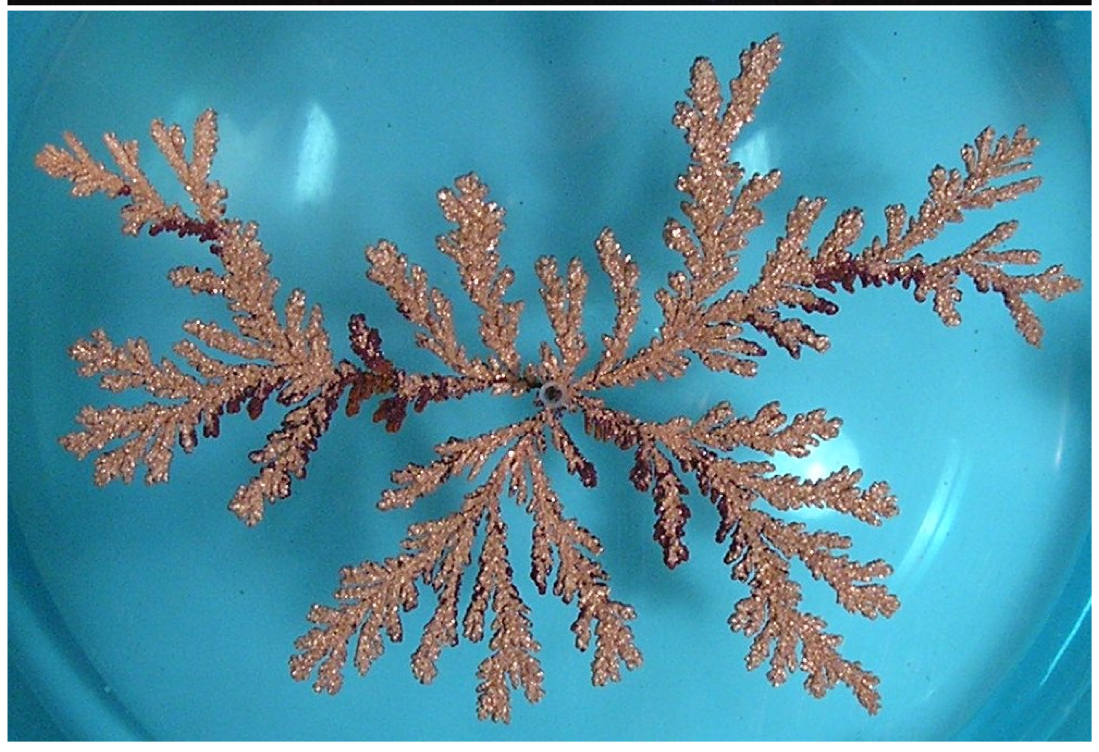
Definitions

Let A be the DLA cluster at some moment in time. In order to construct cluster at the next moment let a new unit disk particle perform a Brownian motion starting far away, until the first time it hits A. Then it sticks to the existing cluster forming the new DLA cluster.

To define DBM-η for $\eta \geq 0$ we let $w(y)$ be the probability density that the new particle hits the cluster at point y (this probability measure is called *harmonic measure*) and then define probability density of attaching as proportional to $w(y)^\eta$.

Note that DLA is equivalent to a DBM-η with $\eta = 1$.

How does It Look Like?



Above: High-voltage dielectric breakdown within a block of plexiglas.

Below: A cluster grown from a copper sulfate solution in an electrodeposition cell.

Many Questions, 1 Answer

The main problem is to describe properties of clusters with a big number of particles and understand what happens when number of particles tends to infinity.

Question 1. *What are the scaling limits of DLA and DBM? In what sense should the scaling limit be understood?*

One of the possible approaches to the problem is to analyze fractal structure, which is clearly observed in experiments and simulations.

Question 2. *What are the fractal dimension and growth rate of DLA and DBM?*

Although this question attracted a lot of researchers and was investigated in numerous papers, only one rigorous theorem is known at the moment. This theorem, due to Kesten [2], gives an upper bound on the growth rate of the DLA. It says that if R is the radius of DLA cluster with n particles, then almost surely

$$R < Cn^{2/3},$$

for n big enough. This means that cluster dimension is at least $3/2$. Simulations suggest that this estimate is not sharp. It is believed that dimension should be ≈ 1.71 for DLA on $\mathbb{R}^2$ and ≈ 1.66 for DLA on $\mathbb{Z}^2$, with the reason for apparent difference still poorly understood.

Kesten's elegant proof of his bound is based on the Beurling's estimate, which bounds harmonic measure of every particle in a size-R connected cluster at every point $w(y) < CR^{-1/2}$.

No non-trivial lower bound on the growth rate is yet known, however, the clusters are expected to be fractal.

In fact, it might happen that the cluster structure should be described not by one scaling parameter, but rather by a family of parameters. Sets with such properties are called *multifractals*.

Question 3. *Do DLA and DBM clusters posses any multifractal properties?*

Growth Upper Bound for DBM

We generalize Kesten's Theorem to DBM-η for $0 \leq \eta < 2$, giving two conceptually different proofs.

Theorem. *Let R be a DBM-η cluster radius after n particles have been attached. Then for $0 \leq \eta < 2$ with probability 1 we have*

$$\limsup_{n\to+\infty} \frac{\log R}{\log n} \leq \frac{2}{4-\eta}.$$

Therefore, dimension of the DBM-η clusters is at least $(4-\eta)/2$.

First proof: Kesten's argument does not automatically carry over to the general case, as new probability density must be estimated. To this effect, in addition to Beurling's estimate we apply discrete analogue [3] of the famous Makarov's Theorem [4], which states that dimension of the harmonic measure equals 1. In discrete setting it means that most of harmonic measure is concentrated at points with measure comparable to R^{-1}, which helps to bound the normalizing constant $\sum_{y\in\partial A} w(y)^\eta$ of attaching probability from below.

Second proof: We develop a new, and somewhat unexpected approach, which exploits multifractal spectra properties in lieu of Beurling's estimates.

Sketch of the 2nd Proof: Multifractal Spectrum

Consider quantities $Z(q) = \sum_{y\in\partial A} w^q(y)$. Standard methods from multifractal analysis suggest that we should have $Z(q) \approx R^{-\tau(q)}$ for some function $\tau(q)$.

One can argue that cluster dimension D is closely related to multifractal structure of cluster harmonic measure. Capacity estimates already exploited in [1] show that we should expect $D = \tau(\eta+2) - \tau(\eta)$. This can be used to give an alternative proof of growth speed upper bound without using Beurling's estimate.

Let σ be such that $\max_y w(y) \approx R^{-\sigma}$. Kesten's argument implies $D \geq 1 - \tau(\eta) + \eta\sigma$. After combining this with the trivial inequality $\sigma \geq \tau(\eta+2)/(\eta+2)$ we get

$$\tau(\eta+2) - \tau(\eta) = D \geq 1 - \tau(\eta) + \frac{\eta\tau(\eta+2)}{(\eta+2)}.$$

Hence, $\tau(\eta+2) \geq (\eta+2)/2$ which, combined with Makarov's Theorem, gives $D \geq (4-\eta)/2$.

Open questions. *1. Can this method be used to improve the Kesten's bound? So far no improvements to his theorem are known.*

2. Can this approach lead to a non-trivial lower bound on the growth rate?

References

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Photos by Bert Hickman and Kevin R Johnson.