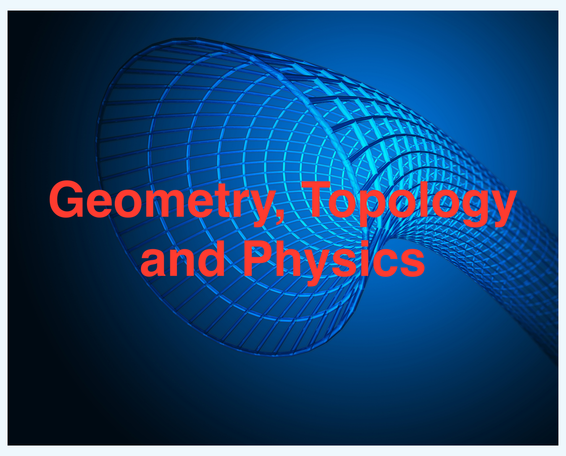

Module traces in pivotal categories

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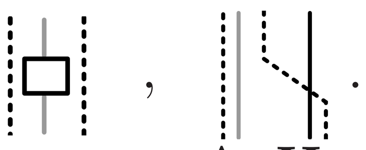
Motivation

The categorical trace plays a central role in the RT invariant [7]. On non-semisimple categories the trace is non-degenerate, making the invariant weaker in this setting. In [6, 4] the modified trace was introduced as a solution to this problem. Indeed, it provides an axiomatization of the trace that allows us to build topological invariants and furthermore, it was proven that it is non-degenerate in the ideal of projective modules over a unimodular Hopf algebra [1, 4]. In this work we try to extend these methods to the non-unimodular case.

Preliminaries

Let $\mathcal{C}$ be a (pivotal) tensor category. A (right) $\mathcal{C}$-module category $\mathcal{M}$ is a linear category together with a functor $\odot\colon \mathcal{M}\times\mathcal{C}\to\mathcal{M}$ satisfying some axioms. A module endofunctor on $\mathcal{M}$ is a structure preserving endofunctor, and its action and structure morphism are represented in the Penrose calculus as



Let G be a group. A **Hopf G-coalgebra** is a family of algebras $H=\{H_g\}_{g\in G}$, together with families of maps called coproduct $\Delta_{gh}\colon H_{gh}\to H_g\otimes H_h$, antipode $S_g\colon H_g\to H_{g^{-1}}$ and counit $\varepsilon\in H_1^*$. It is called pivotal if there exists a group-like element $\boldsymbol{g}\in\prod_{h\in G}H_h$ for which $S_{h^{-1}}S_h(a)=\boldsymbol{g}_h a\boldsymbol{g}_h^{-1}$. An (right) **integral** for a pivotal Hopf G-coalgebra with pivot $(\boldsymbol{g}_h)_{h\in G}$ is a family of linear forms $\hat{\boldsymbol{\mu}}_h\in H_h^*$ such that

$$(\hat{\boldsymbol{\mu}}_x\otimes\boldsymbol{g}_y)\circ\Delta_{x,y}(a)=\hat{\boldsymbol{\mu}}_{xy}(a)1_y\,,\quad a\in H_{xy}.$$

The **modulus** is an algebra map $\boldsymbol{\alpha}\colon H_1\to\Bbbk$, which is an obstruction to the integral of the dual Hopf algebra to be two sided. This induces a module endofunctor $R(\boldsymbol{\alpha})_*$ on projective modules by tensoring with the one dimensional representation twisted by $\boldsymbol{\alpha}$.

Module trace

Let G be a group, $\mathcal{C}$ a pivotal $\Bbbk$-tensor G-category, $\mathcal{M}$ a (right) $\mathcal{C}$-module category and $\Sigma\colon\mathcal{M}\to\mathcal{M}$ a module functor. A (right) module trace on $(\mathcal{M},\Sigma)$ is a family of linear maps

$$\{t_M\colon\mathcal{M}(M,\Sigma M)\to\Bbbk\}_{M\in\mathcal{M}}$$

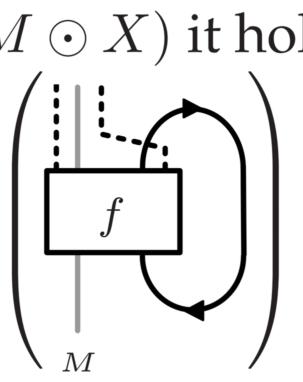
satisfying the following two conditions:

Cyclicity: for all objects $M,N\in\mathcal{M}$ and morphisms $f\colon M\to N$, $g\colon N\to\Sigma M$ it holds that

$$t_M(gf)=t_N\big((\Sigma f)g\big).$$

Partial trace property: For all $M\in\mathcal{M}$, $X\in\mathcal{C}$ and $f\colon M\odot X\to\Sigma(M\odot X)$ it holds

$$t_{M\odot X}(f)=t_M$$



The modified trace of [6, 4] can be recovered as the special case when $\mathcal{M}$ is a tensor ideal of $\mathcal{C}$ and Σ the identity functor.

Main results

Theorem 1. *Let $(H,\boldsymbol{g})$ be a finite type pivotal Hopf G-coalgebra and $\boldsymbol{\alpha}$ the modulus of H_1. Then the space of module traces on $(H\text{-pmod},R(\boldsymbol{\alpha})_*)$ is equal to the space of integrals, and thus one-dimensional. Furthermore, the module trace t is non-degenerate and determined by*

$$t_{H_x}(f)=\hat{\boldsymbol{\mu}}_x\big(f(1_x)\big)$$
$$f\in\mathrm{Hom}_{H_x}\big(H_x,R(\boldsymbol{\alpha})_*(H_x)\big),\quad x\in G\,.$$

Corollary 2. *Let $(H,\boldsymbol{g})$ be a finite type pivotal Hopf G-coalgebra and A be a unimodular sub Hopf G-coalgebra in H such that $\boldsymbol{g}\in A$. Then, there is a right modified trace t on the tensor ideal*

$$\mathcal{I}_A=\{V\in H\text{-mod}\,|\,\mathrm{Res}_A(V)\in A\text{-pmod}\}$$

given by the pull-back

$$t_M(f)=t'_{\mathrm{Res}(M)}\big(\mathrm{Res}_A(f)\big)\,,$$

where Res *is the restriction functor and t' is the trace on A-pmod.*

Modified trace on full quantum $\mathfrak{sl}_2$

Let $p\in\mathbb{N}_{\geq 3}$ and $q=e^{\frac{\pi i}{p}}$. Consider the following quotients of $U_q\mathfrak{sl}_2$, $U_{(y,z,x)}=U_q\mathfrak{sl}_2/\langle F^p-y,K^p-z,E^p-x\rangle$. It is a Hopf G-coalgebra for $G=\mathbb{C}\rtimes\mathbb{C}^\times\ltimes\mathbb{C}$. It is unimodular and its integral is given by

$$\hat{\boldsymbol{\mu}}_{(y,z,x)}(E^lF^jK^n)=z^{1+n}\delta_{l,p-1}\delta_{j,p-1}\delta_{n,0}\,,$$

where we have chosen K^{1+np} as a pivot.

It is known [5] that $U=\{U_{(y,z,x)}\}_{(y,z,x)\in G}$ is generically semisimple, that is, $U_{(y,z,x)}$ is semisimple if and only if $(q-q^{-1})^{2p}xy-z-z^{-1}\neq\pm2$.

To compute the modified trace, it is sufficient to determine its value on endomorphisms of projective indecomposables.

In the semisimple case, projective indecomposable $U_{y,z,x}$-modules are all p-dimensional. We will denote them $V_{y,\lambda,x}$ where $\lambda^p=z$. This module is associated to the primitive idempotent

$$I_{c,\lambda}=\frac{\psi(C_q)}{p\psi(c)}\left(\sum_{i=0}^{p-1}\lambda^{-i}K^i\right),$$

where C_q is the Casimir element, which on $V_{y,z,a}$ acts as $c=ay+(q^{-1}\lambda+\lambda^{-1}q)(q-q^{-1})^{-2}$, and ψ is a polynomial obtained from the characteristic polynomial of C_q by removing the factors associated to c. Straightforward computation yields

$$t(\mathrm{Id}_{V_{y,\lambda,a}})=\frac{z^{1+n}}{p\psi(c)}$$

Notice that the cases $x,y=0$, $z=\pm1$ correspond to the representation theory of the small quantum group. The modified trace was computed in [1] using similar methods. And so, the only remaining computations correspond to the quotient $U_{(y,z,x)}$ for $(q-q^{-1})^{2p}xy-z-z^{-1}=\pm2$ and at least one of x,y non-zero. In this case projective indecomposables are given by $V_{y,\lambda,a}$ modules, corresponding to simple roots of the characteristic polynomial of C_q, and their self-extensions $V^2_{y,\lambda,a}$, corresponding to double roots. For simple roots computations are similar to the semisimple case. For double roots, computing primitive idempotents is somewhat more complex, and furthermore the space of endomorphisms of $V^2_{y,\lambda,a}$ is two dimensional.

In view of Corollary 2, we can use this computations to recover the computation of the trace on the projective ideal of the unrolled quantum group in [3], but now in a merely algebraic way, as opposed to the categorical approach of said paper.

Borel subalgebra

Let $\mathcal{B}_q$ be the subalgebra of $U_q\mathfrak{sl}_2$ generated by E and K. As before we can analyze this using the family of quotients $\mathcal{B}_q^{z,x}=\mathcal{B}_q/(K^p-z,E^p-x)$ for $z\in\mathbb{C}^\times$, $x\in\mathbb{C}$. The integral of this algebra is given by

$$\hat{\boldsymbol{\mu}}_{(z,x)}(E^iK^j)=z^{1+n}\delta_{i,p-1}\delta_{j,0}.$$

Unlike $U_q\mathfrak{sl}_2$ however this algebra is not unimodular. Its modulus is determined by $\boldsymbol{\alpha}(E)=0\,,\ \boldsymbol{\alpha}(K)=q^2$.

A complete set of primitive idempotents is given by

$$e_s=\frac{1}{p}\sum_{i=0}^{p-1}z^{-i/p}q^{2is}K^i\,,\ s=0,\dots,p-1,$$

and we associate to each the projective indecomposable module $V_s=\mathcal{B}_q^{z,x}e_s$. The induced functor $R(\boldsymbol{\alpha})_*$ is determined by $R(\boldsymbol{\alpha})_*(V_s)\cong V_{s-1}$.

Next, we determine the hom spaces $\mathrm{Hom}(V_s,V_{s-1})\cong e_s\mathcal{B}_q^{z,x}e_{s-1}$. This is one dimensional and generated by $r_s^{x,z}\colon e_s\to E^{p-1}e_{s-1}$.

Finally, the module traces can be computed to be

$$t_{V_s}(r_s^{x,z})=\hat{\boldsymbol{\mu}}_{z,x}(E^{p-1}e_s)=\frac{z^{1+n}}{p}.$$

Future directions

- We think that this construction might prove relevant in the study of catgories of representations of Lie algebras in positive characteristic.
- We expect to extend the invariant of B-tangles of [2] to the non-semisimple case.

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