The Gibbs distribution

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Let $\Omega$ be a finite configuration space of a physical system. The energy of the configuration $\omega \in \Omega$ is $H(\omega)$.

Example: Ising model on a finite graph $(V, E)$:

\[
\Omega = \{-1, +1\}^V, \quad H(\omega) := -\sum_{\{x, y\} \in E} \omega_x \omega_y.
\]

Gibbs distribution: probability measure $\mu$ on $\Omega$ that has fixed energy and maximal entropy.
Entropy

Quantify the disorder or “mixed-up-ness” [Gibbs, 1903] of the measure $\mu$. Informally:

$$S = \log |\{\text{microstates giving rise to the observed macrostate}\}|.$$ 

If $\mu$ is the uniform distribution on $\Omega$ (all configurations of $\Omega$ are equally likely) then

$$S(\mu) = \log |\Omega|.$$ 

Why log?

Entropy should be extensive. Putting two systems, 1 and 2, together amounts to $\Omega = \Omega_1 \times \Omega_2$ and $\mu = \mu_1 \otimes \mu_2$. Hence, $S(\mu) = S(\mu_1) + S(\mu_2)$. 
How to define $S$ for general $\mu$? Simple model: put $n$ balls into the boxes of $\Omega$:

- **microstate** = individual balls’ locations
- **macrostate** = number of balls in each box.

More formally: To $b = (b_1, b_2, \ldots, b_n) \in \Omega^n$ (individual balls’ locations) assign $N_\omega(b) := |\{1 \leq i \leq n : b_i = \omega\}|$ (number of balls in box $\omega$).

Then $b$ is the microstate and $\mathbf{N}(b) = (N_\omega(b))_{\omega \in \Omega}$ is the associated macrostate.

For $\mathbf{N} \in \mathbb{N}^\Omega$ define

$$W(\mathbf{N}) := |\{b \in \Omega^n : \mathbf{N}(b) = \mathbf{N}\}| .$$

**Exercice.** Suppose that $b_1, b_2, \ldots$ are i.i.d. random variables in $\Omega$ with law $\mu$. Then, almost surely as $n \to \infty$,

$$\frac{1}{n} \log W(\mathbf{N}(b)) \to S(\mu) := - \sum_{\omega \in \Omega} \mu(\omega) \log \mu(\omega) .$$
The quantity

\[ S(\mu) = - \sum_{\omega \in \Omega} \mu(\omega) \log \mu(\omega) \]

is the (Boltzmann-Gibbs-Maxwell-Shannon) entropy of the probability measure \( \mu \).

Exercise.

- \( 0 \leq S(\mu) \leq \log |\Omega| \).
- \( S(\mu) = 0 \) if and only if \( \mu = \delta_{\omega_0} \) for some \( \omega_0 \in \Omega \).
- \( S(\mu) = \log |\Omega| \) if and only if \( \mu \) is uniform on \( \Omega \).
The Gibbs measure

Define the energy of the distribution $\mu$ as

$$U(\mu) := \sum_{\omega \in \Omega} H(\omega) \mu(\omega).$$

Maximize the entropy $S(\mu)$ under fixed energy $U(\mu)$.

Exercise.

$$\mu(\omega) = \frac{1}{Z} e^{-\beta H(\omega)}, \quad Z := \sum_{\omega \in \Omega} e^{-\beta H(\omega)}.$$  

- $\beta$ is a Lagrange multiplier with the interpretation of the inverse temperature, $\beta = 1/T$.
- $Z$ is the partition function.
- The (Helmholtz) free energy is $F := -T \log Z$. We obtain the thermodynamic relation

$$F = U - TS.$$
In most interesting applications, $\Omega$ is infinite, and the notion of Gibbs measure is much more subtle. It is best formulated in the framework of spin systems: A probability space $(S, \lambda)$ is assigned to each site of a lattice $L$. The configuration space is $\Omega = S^L$ with configuration $\omega = (\omega_x)_{x \in L}$.

**Notations:** Let $\Lambda \subset L$.

- $\omega_\Lambda = (\omega_x)_{x \in \Lambda}$ and $\omega = \omega_\Lambda \omega_{\Lambda^c}$.
- $\lambda(d\omega_\Lambda) = \prod_{x \in \Lambda} \lambda(d\omega_x)$.

For each finite $A \subset L$, introduce a potential $\Phi_A$ that depends only on $\omega_A$. Defining $H(\omega) := \sum_{A \subset L} \Phi_A(\omega)$, we want to define the Gibbs measure

$$\mu(d\omega) = \frac{1}{Z} e^{-\beta H(\omega)} \lambda(d\omega), \quad Z := \int \lambda(d\omega) e^{-\beta H(\omega)}$$

Only makes sense for finite $L$. 
The Dobrushin-Lanford-Ruelle (DLR) equation

Let $\Lambda \subset L$ be finite. For a boundary condition $\eta \in S^L$ define the conditional energy

$$H(\omega_\Lambda | \eta_{\Lambda^c}) := \sum_{A \subset L : A \cap \Lambda \neq \emptyset} \Phi_A(\omega_\Lambda \eta_{\Lambda^c})$$

and the conditional partition function

$$Z_\Lambda^\eta := \int \lambda(d\omega_\Lambda) e^{-\beta H(\omega_\Lambda | \eta_{\Lambda^c})}.$$
Exercise. For finite $L$ and bounded $f : \Omega \to \mathbb{R}$ we have

$$\int \mu(d\omega) f(\omega) = \int \mu(d\eta) \frac{1}{Z_\Lambda} \int \lambda(d\omega_\Lambda) e^{-\beta H(\omega_\Lambda|\eta_{\Lambda^c})} f(\omega_\Lambda \eta_{\Lambda^c}). \quad \text{(DLR)}$$

For infinite $L$, take (DLR) as the definition of a Gibbs measure:

**Definition.** A probability measure $\mu$ satisfying (DLR) for all finite $\Lambda \subset L$ and $f$ of bounded support is called a Gibbs measure associated with $(\Phi_A)$.

**Existence** is easy under a general locality assumption on $(\Phi_A)$ (weak compactness argument).

**Uniqueness** is delicate and in general wrong: coexistence of phases (dependence on boundary conditions).