CFTs at large non-Abelian charge:

from superfluids to the critical $O(N)$ models

Gabriel Cuomo

based on
GC, A. Esposito, E. Gendy, A. Khmelnitsky, A. Monin, R. Rattazzi 2005.12924
& work in progress

Short talk at the 7th SwissMAP General Meeting

EPFL

SwissMAP
The Mathematics of Physics
National Centre of Competence in Research

Overview

1. Large charge operators in CFT
2. Intermezzo: general properties of superfluids
3. Application: the superfluid phase of 3d CFTs

Large charge operators in CFT

Conformal field theories (CFTs) and their study

Why? QFTs $\sim$ conformal between very separated energy scales $\Lambda_1 \ll E \ll \Lambda_2$; also AdS/CFT, RG, critical points...

What? Observables in conformal field theories (CFTs):

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \sim \frac{\delta_{ij}}{x^{2\Delta_i}}, \qquad \langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle \sim \lambda_{ijk}.$$

How?

Very few perturbative CFTs:

- ε-expansion
- large N

Non-perturbative methods:

- lattice simulations
- conformal bootstrap

In $d > 2$ most useful only for low twist operators: $\Delta - J \sim \mathcal{O}(1)$.

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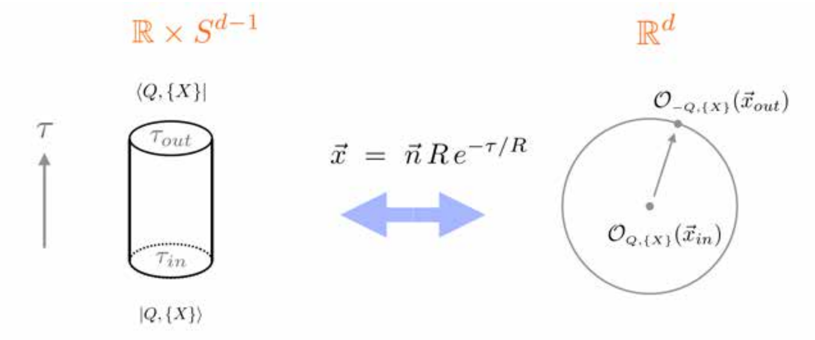
Large charge expansion in CFT

- d-dimensional CFT invariant under internal G.
- Lowest dimension operator at given charge Q : $\mathcal{O}_{Q,\{X\}}(x)$.
- For $Q \gg 1$ compute semiclassically:

$$\langle \mathcal{O}^{\dagger}_{Q',\{X'\}}(x_{out}) \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)}_{\text{light operators}} \mathcal{O}_{Q,\{X\}}(x_{in})\rangle.$$

S. Hellerman, D. Orlando, S. Reffert, M. Watanabe 1505.01537
A. Monin, D. Pirtskhalava, R. Rattazzi, F. Seibold 1611.02912

State-operator correspondence



$$H|Q, \{X\}\rangle = E_Q|Q, \{X\}\rangle, \qquad E_Q = \Delta_0(Q)/R.$$

Large charge state

State-operator correspondence:

$$\langle \mathcal{O}^\dagger_{Q',\{X'\}}(x_{out})\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\mathcal{O}_{Q,\{X\}}(x_{in})\rangle$$
$$\downarrow$$
$$\langle Q'|\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)|Q\rangle$$

Properties of $|Q\rangle$:

- charge density: $j_0 \sim Q/R^{d-1} \propto \mu^{d-1} \gg \frac{1}{R^{d-1}}$.
- condensed matter phase on the cylinder:
  $\to$ *effectively* nonlinearly realizes $SO(d+1,1)\times G$.
- Simplest option: $|Q\rangle =$"conformal superfluid".

Intermezzo: general properties of superfluids

Finite density states

Finite density states nonlinearly realize spacetime symmetries:

- hydrodynamic modes are given by Goldstone Bosons.

A. Nicolis, R. Penco, F. Piazza, R. Rattazzi 1501.03845

Ex. $U(1)$ superfluid: $ISO(d-1,1) \times U(1)_Q \to ISO(d-1) \times \underbrace{H'}_{=H-\mu\widehat{Q}}$:

- $\mathcal{L} = P(\partial\chi), \qquad (\partial\chi) = \sqrt{\partial_\mu\chi\partial^\mu\chi},$
- $\widehat{Q} : \chi(x) \to \chi(x) + c,$
- $\chi(x) = \mu t + \pi(x) \leftarrow$ hydrodynamic mode.

D. Son 0204199

Superfluids and Goldstone bosons (GBs)

Relativistic system at finite density for a given charge Q:

$$\bar{H}|\mu\rangle = (H + \mu Q)|\mu\rangle = 0.$$

- $\not{H}$, $\not{Q}$ but $\bar{H}$ unbroken (*nonrelativistic* Hamiltonian).
- $\not{Q}_a$ commuting with $Q \implies$ gapless GBs.
- $\not{Q}_a$ non commuting with $Q \implies$ gapped GBs:

$$[Q, Q_a^{\pm}] = \pm q_a Q \quad \implies \quad \omega(\mathbf{0}) = q_a \mu.$$

- "Radial" modes *generically* gapped at least at μ.

R. Lange 1965; A. Nicolis, F. Piazza 1204.1570

Which EFT when μ and the strong coupling scale coincide?

The nonrelativistic EFT of gapped Goldstones

- All (gapless and gapped) GBs are free when at rest:

$$\lim_{\boldsymbol{p}\to 0} \mathcal{M}\left(\alpha \to \beta + \pi(\boldsymbol{p})\right) = 0$$

T. Brauner, M. F. Jakobsen 1709.01251

EFT describes soft gapless GBs and slow gapped GBs in an expansion in $\boldsymbol{p}/\mu$.

- Gapped GBs decay/annihilation into hard gapless modes: described *inclusively* via absorbitive (imaginary) terms.
- Construction is made systematic via a rearrangement of the CCWZ construction.

Callan, Coleman, Wess, Zumino 1969

Example: $SU(2) \to \emptyset$ at finite Q_3 density

- $SU(2)$ parametrized with a real and a complex non-relativistic field:

$$\Omega = e^{i\chi Q_3} e^{i\alpha \frac{Q_+}{2} + c.c.}, \qquad \chi = \mu t + \pi_3\,, \quad \alpha = \pi e^{-i\mu t} \quad \partial\pi_3, \partial\pi \ll \mu\,.$$

- $SU(2)$ invariants:

$$-i\Omega\partial_\mu\Omega = D_\mu\chi Q_3 + D_\mu\alpha Q_+/2 + c.c.$$

- Higher derivatives built with the *non-relativistic derivative*:

$$\partial_\mu \to \hat{\partial}_\mu = \partial_\mu + iD_\mu\chi[Q_3, \cdot] \qquad \Longrightarrow \qquad \hat{\partial}_0 D_\mu\alpha \simeq (\partial_0 + i\mu)\partial_\mu\alpha = \partial_0\partial_\mu\pi\,.$$

- The action ($n_\mu = D_\mu\chi/D\chi \simeq \delta^0_\mu$):

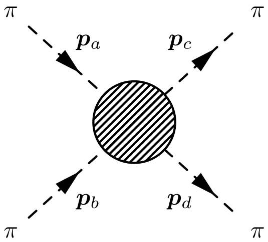
$$\mathcal{L}_{\text{eff}} = c^{(1)}\mu^3\left(n^\mu D_\mu\chi - \mu\right) + c_1^{(2)}\mu^2\left(n^\mu D_\mu\chi - \mu\right)^2 + c_2^{(2)}\left|n^\mu D_\mu\pi\right|^2 + 4c^{(1)}c_m D_\mu\pi^*\left[\eta^{\mu\nu} - n^\mu n^\nu\right]D_\nu\pi + \mathcal{O}\left(\hat{\partial}^3/\mu^3\right)$$

Example: $SU(2) \to \emptyset$ at finite Q_3 density

- Gapped GB dispersion relation:

$$E_p = \mu + c_m \frac{\boldsymbol{p}^2}{2\mu} \quad \Longrightarrow \quad \Gamma_p = -2\mathrm{Im}\left[E_p\right] = -\mathrm{Im}\left[c_m\right] \frac{\boldsymbol{p}^2}{\mu}$$

- Gapped GB scattering



$$= \frac{i}{4c^{(1)}\mu^4} \left\{ \frac{(\boldsymbol{p}_a^2 - \boldsymbol{p}_c^2)^2}{(\boldsymbol{p}_a - \boldsymbol{p}_c)^2} + \frac{(\boldsymbol{p}_a^2 - \boldsymbol{p}_d^2)^2}{(\boldsymbol{p}_a - \boldsymbol{p}_d)^2} + (2c_m - 3)\left(\boldsymbol{p}_a + \boldsymbol{p}_b\right)^2 + 2(1 - c_m)\left(\boldsymbol{p}_a^2 + \boldsymbol{p}_b^2\right) \right\}$$

- Results can be consistently matched to UV complete models.

Application: the superfluid phase of 3d CFTs

The $3d$ $O(2)$ model

- Single Goldstone field $\chi = \mu t + \pi(x)$:

$$\mathcal{L} = c(\partial\chi)^3\,, \qquad \frac{Q}{4\pi R^2} = 3c\,\mu^2\,.$$

- Ground state energy:

$$\Delta_0(Q) = RE_Q = \alpha Q^{\frac{3}{2}} + \beta\,Q^{\frac{1}{2}} - 0.0937 + \dots\,.$$

Result verified by Monte-Carlo.

D. Banerjee, S. Chandrasekharan, D. Orlando 1707.00711

- CFT spectrum described by *phonon* Fock space:

$$\omega_\ell^2 = \frac{1}{2} \times \frac{\ell(\ell+1)}{R^2}\,,$$

$$\Delta(Q, \{n_\ell\}) = \Delta_0(Q) + \sum n_\ell R\omega_\ell\,.$$

The $3d$ $O(4)$ model

- ε-expansions suggests $O(4) \to O(2)$ superfluid phase.
- Spectrum consists of 1 gapless and 2 gap-μ GBs.
- The unbroken $O(2)$ forbids the mixing between gapped GBs and radial modes.
- Operators with lowest dimension in generic irrep.s correspond to n-gapped Goldstone states:

$$(Q_L, Q_R) = \left(\frac{Q}{2} - n, \frac{Q}{2}\right), \qquad 0 \leq n/Q \ll 1,$$

$$\Delta = \Delta_0(Q) + \gamma \frac{n}{\sqrt{Q}}, \qquad \ell \subset \underbrace{1 \otimes 1 \ldots \otimes 1}_{n-times}.$$

Results potentially testable by Monte-Carlo simulations.

Summary and outlook

Summary:

- The state-operator correspondence and the superfluid EFT allow to study correlators of $\mathcal{O}_Q(x)$ for $Q \gg 1$.
- The EFT for non-Abelian superfluids describes slow gapped GBs and soft gapless ones.
- Spectrum of the critical $O(N)$ model at large Q.

Outlook:

- other phases: Fermi liquids? Reissner-Nordström BH in AdS?
- other large quantum numbers: J, Δ.
- relation to the bootstrap.

Thank you for the attention

Backup slides

$SU(2)$-invariant $3d$ CFT

- $SU(2)$ fully broken by the Q_3 density.
- Spectrum includes 1 gapless and 1 gap $\omega(\mathbf{0}) = \mu$ Goldstones.
- Properties of lowest energy states determined by the gapless GB only as if $SU(2) \to U(1)$. E.g. ground state energy:

$$\Delta_0(Q) = \alpha_1 Q_3^{\frac{3}{2}} + \alpha_2 Q_3^{\frac{1}{2}} - 0.0937 + \ldots\,, \qquad \vec{Q}^2 = Q_3(Q_3+1)\,.$$

- At finite volume decay/annihilation of gapped GBs translate into mixing with a *discrete* number of states outside the EFT.
- We can describe only *inclusive* observables, aka insensitive to the discreteness of the spectrum. ($\sim$ Higgs decay in KK theories).

G. F. Giudice, R. Rattazzi, J. D. Wells 0002178

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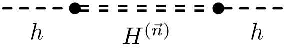
Invitation: Higgs-graviscalar mixing in KK theories

- Higgs graviscalar mixing in $d+\delta$-dimensional KK theories:

$$\Sigma(p^2) =$$



$$= \frac{m_{mix}^{4-\delta}}{(2\pi R)^\delta} \sum_{\vec{n}\in\mathbb{Z}^\delta} \frac{-1}{p^2 - \vec{n}^2/R^2 + i\varepsilon}$$

$$\hat{A}(\omega) = \int dt\, e^{i\omega t} \langle h, t | h, 0 \rangle \simeq \frac{i}{\omega - m_h + \frac{1}{2m_h}\Sigma(\omega^2) + i\varepsilon}$$

- Model detector spread via Lorentzian

$$\widehat{A}_{mes}(\omega_0) = \int d\omega \frac{L/\pi}{(\omega-\omega_0)^2 + L^2} \hat{A}(\omega) = \hat{A}(\omega + iL)$$

- When the resolution is much larger than the level separation, $L \gg n_{m_h}^{-1} \propto R^{-\delta}$, we can use a continuous approximation.

Invitation: Higgs-graviscalar mixing in KK theories

- Continuous approximation:

$$\Sigma(p^2 + iL) \simeq \frac{m_{mix}^{4-\delta}}{(2\pi)^\delta} \int d^\delta q_\perp \frac{-1}{p^2 - q_\perp^2 + iL}$$

- If resolution L is much smaller than any lab scale:

$$\Gamma_h = \frac{1}{m_h} \mathrm{Im}[\Sigma(m_h^2 + iL)] \simeq \frac{\pi}{2} \frac{\Omega_{\delta-1}}{(2\pi)^\delta} \frac{m_{mix}^{4-\delta}}{m_h^{3-\delta}}$$

- Detector effectively measures a resonance

$$\widehat{A}_{mes}(\omega) \simeq \frac{i}{\omega - m_h - i\frac{\Gamma_h}{2}}$$

$SU(2)$-invariant 3d CFT: gapped Goldstone resonance

- Non-Abelian current correlator:

$$\langle Q|T\left\{J^0_+(t,\hat{n}_2)J^0_-(0,\hat{n}_1)\right\}|Q\rangle = \int d\Delta \sum_{\ell=0}^{\infty}\rho^{(\ell)}(\Delta)g_{\Delta,\ell}(t,\hat{n}_2\cdot\hat{n}_1)\,,$$

$$\rho^{(\ell)}(\Delta) = \sum_A |\lambda_A|^2\delta(\Delta-\Delta_A)\,.$$

- *Smeared* spectral function is regular:

$$\hat{\rho}^{(\ell)}(\Delta) \equiv \int d\tilde{\Delta}\frac{L/\pi}{(\tilde{\Delta}-\Delta)^2+L^2}\rho^{(\ell)}(\tilde{\Delta})\,, \qquad L \gg 1/n^{(\ell)}_{\Delta_Q}\,,$$

$$\ell \lesssim n^{(\ell)}_{\Delta_Q} \lesssim \exp\left[b\,Q^{1/3}\right]\,.$$

- EFT predicts

$$\hat{\rho}^{(\ell)}(\Delta) = \frac{Q(2\ell+1)}{8\pi^2R^4}\frac{\Gamma_\ell/(2\pi)}{(\Delta-\Delta_Q-\epsilon_\ell)^2+\Gamma_\ell^2/4}\,,$$

$$\epsilon_\ell = \mathrm{Re}[c_m]\frac{\ell(\ell+1)}{2R\mu}\,, \qquad \Gamma_\ell = -\mathrm{Im}[c_m]\frac{\ell(\ell+1)}{R\mu}\,.$$

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