BPS counting with Exponential Networks

Pietro Longhi
ETH Zürich

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Based on:

- 20xx.xxxxx
- 1910.05296,
- 1811.02875

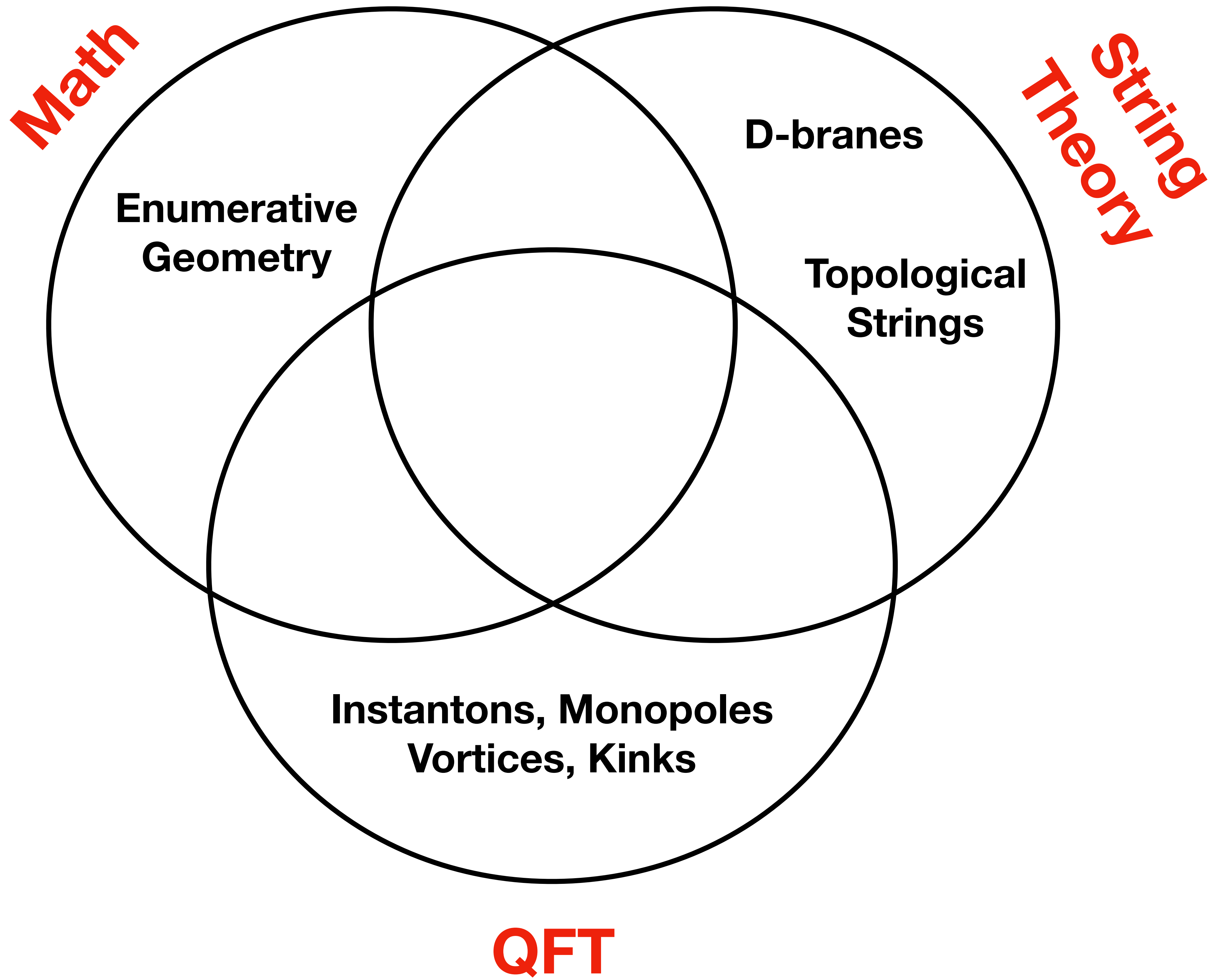
With:
S. Banerjee (Köln)
M. Romo (Tsinghua)

1. Introduction to the Math-Physics setup

2. Exponential Networks

3. Applications

What is… “BPS counting,, ?



M theory on $\mathbb{R}^4 \times S^1 \times X$

- X **(toric) CY 3-fold**

BPS States

- **M2 brane on $\mathbb{R} \times C_2$**
- **M5 brane on $\mathbb{R} \times S^1 \times C_4$**

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Geometric engineering (QFT)

- **5d gauge theory** $T[X]$

BPS States

- **El. particles** $\mathbb{R}$**,** $M = Vol(C_2)$
- **Mag. string** $\mathbb{R} \times S^1$**,** $T = Vol(C_4)$

Goal: **Develop a systematic framework to compute the spectrum of BPS states, encoded by $\Omega(\gamma) \in \mathbb{Z}$;**
[γ = (D6, D4, D2, D0) charge]

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Examples:

- $\Omega(k\,\mathbf{D0}) = -\chi(X) \quad \forall k$
- **If $\nexists$ compact $C_4 \subset X$**

  $\Omega(C_2 + k\,\mathbf{D0}) = n_0^{C_2} \quad \forall k$ **(genus-0 Gopakumar-Vafa inv.)**

- **More generally, including $C_4 \subset X$**

  $\Omega(C_4 + C_2 + k\,\mathbf{D0}) = VW_{C_4,C_2,k}$ **(Vafa-Witten inv.)**

- **Using Wall-Crossing formulae**

  $\Omega(N\,\mathbf{D6} + C_4 + C_2 + k\,\mathbf{D0})$ **((higher rk) Donaldson-Thomas inv.)**

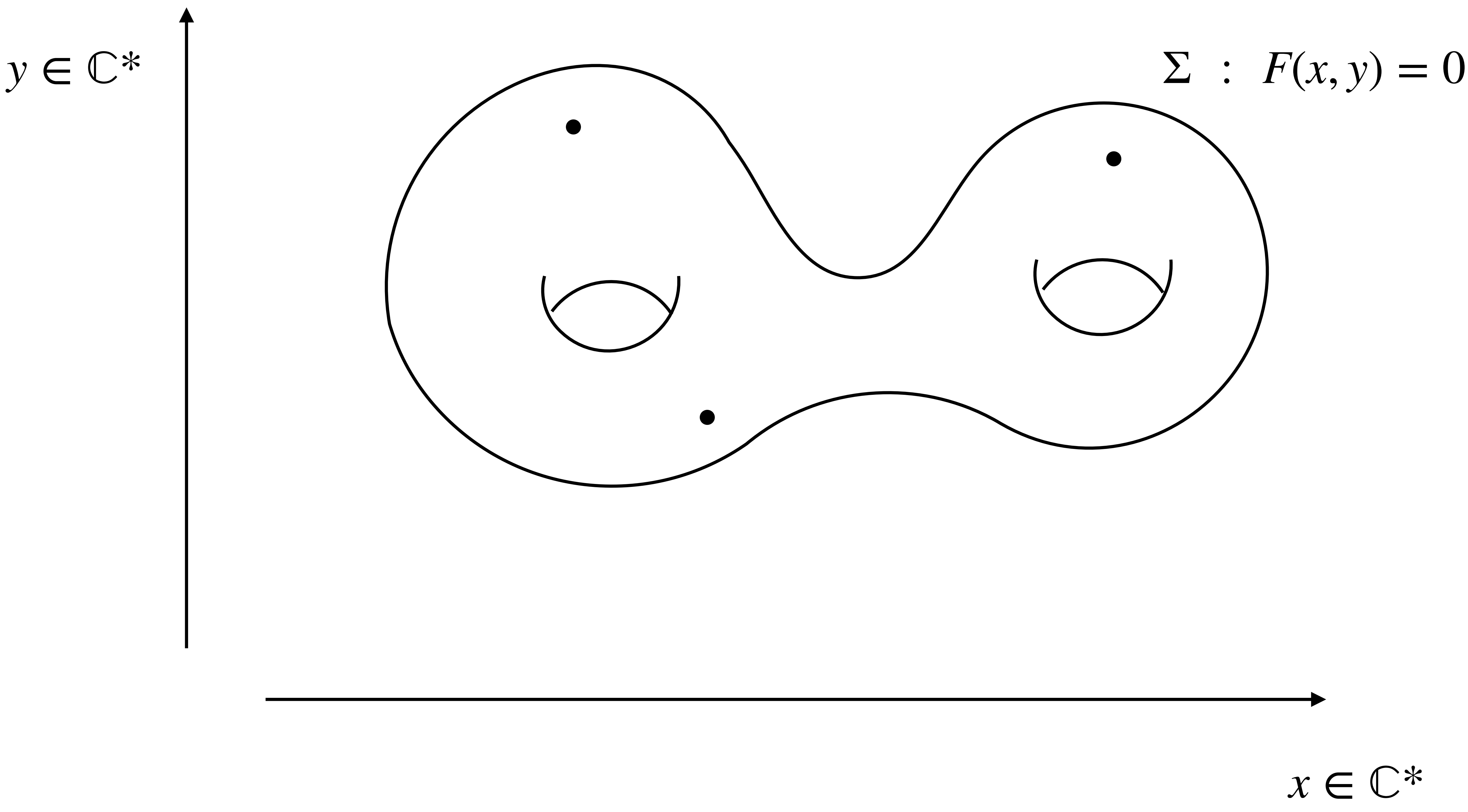
Approach: Exponential Networks

A very useful characterization of these BPS states was given by [Klemm-Lerche-Mayr-Vafa-Warner ’96]

- **BPS states on** $X \quad \Leftrightarrow \quad$ **BPS states on the “mirror,,** Y

$$uv = F(x, y) \quad \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$$

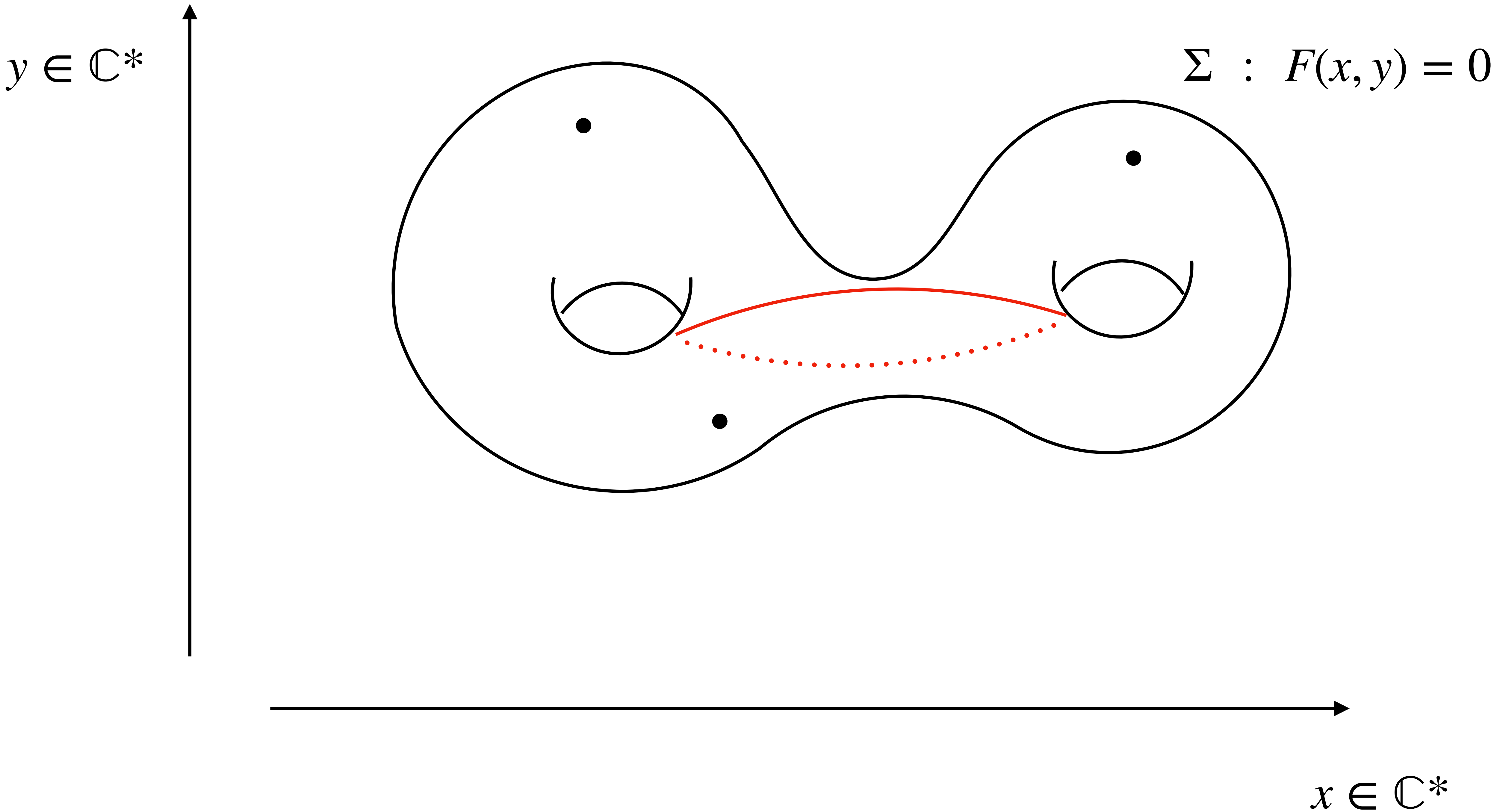
- **Geodesics on the Riemann surface** $\Sigma \;:\; F(x, y) = 0$



$y \in \mathbb{C}^*$
$\Sigma \; : \; F(x, y) = 0$
$x \in \mathbb{C}^*$

$\lambda = \log y \; d\log x$

BPS Geodesics: $\gamma(t) : S^1 \to \Sigma$ **such that** $\arg \lambda(t) = \vartheta$ **const.**



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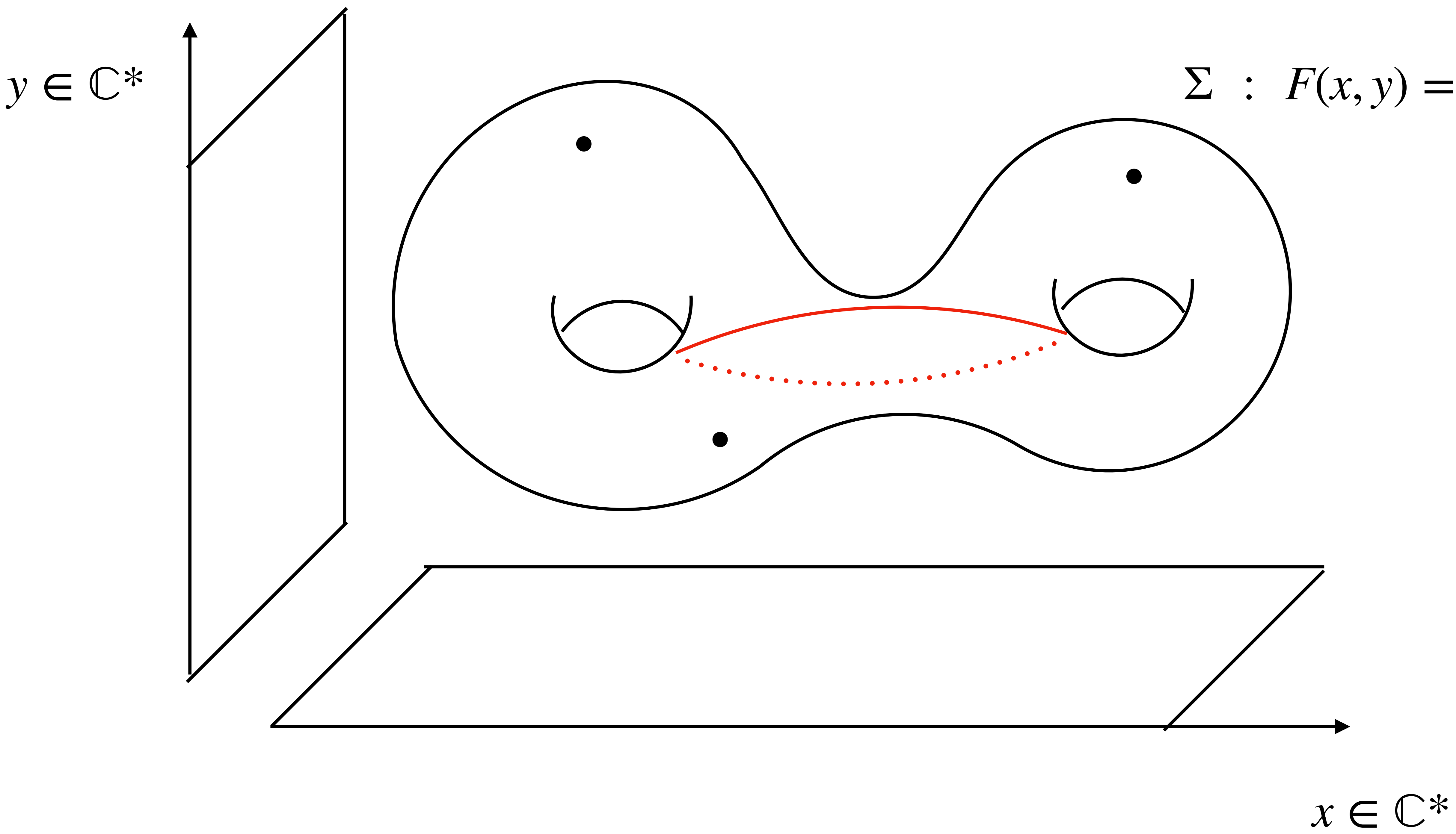

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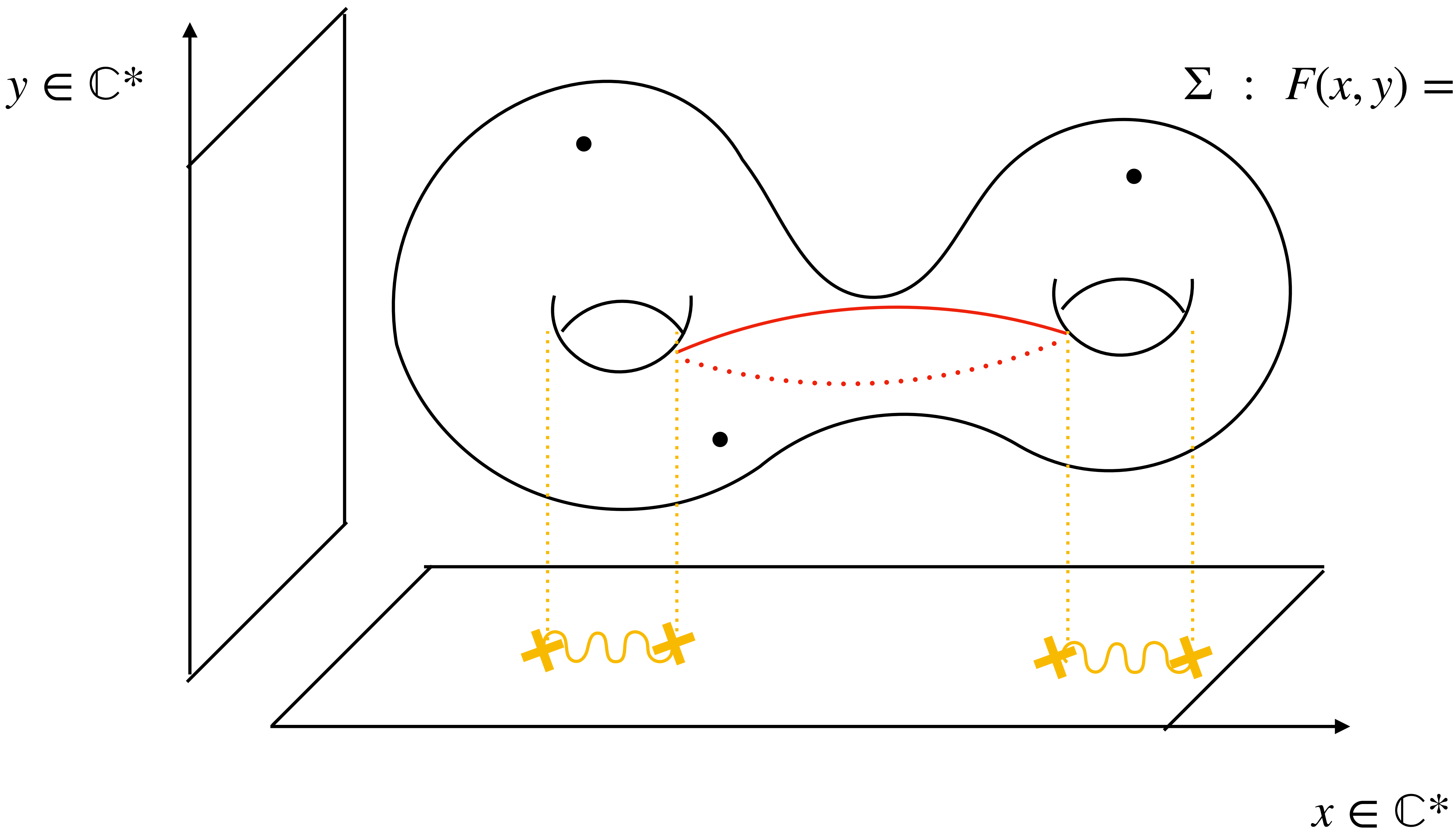

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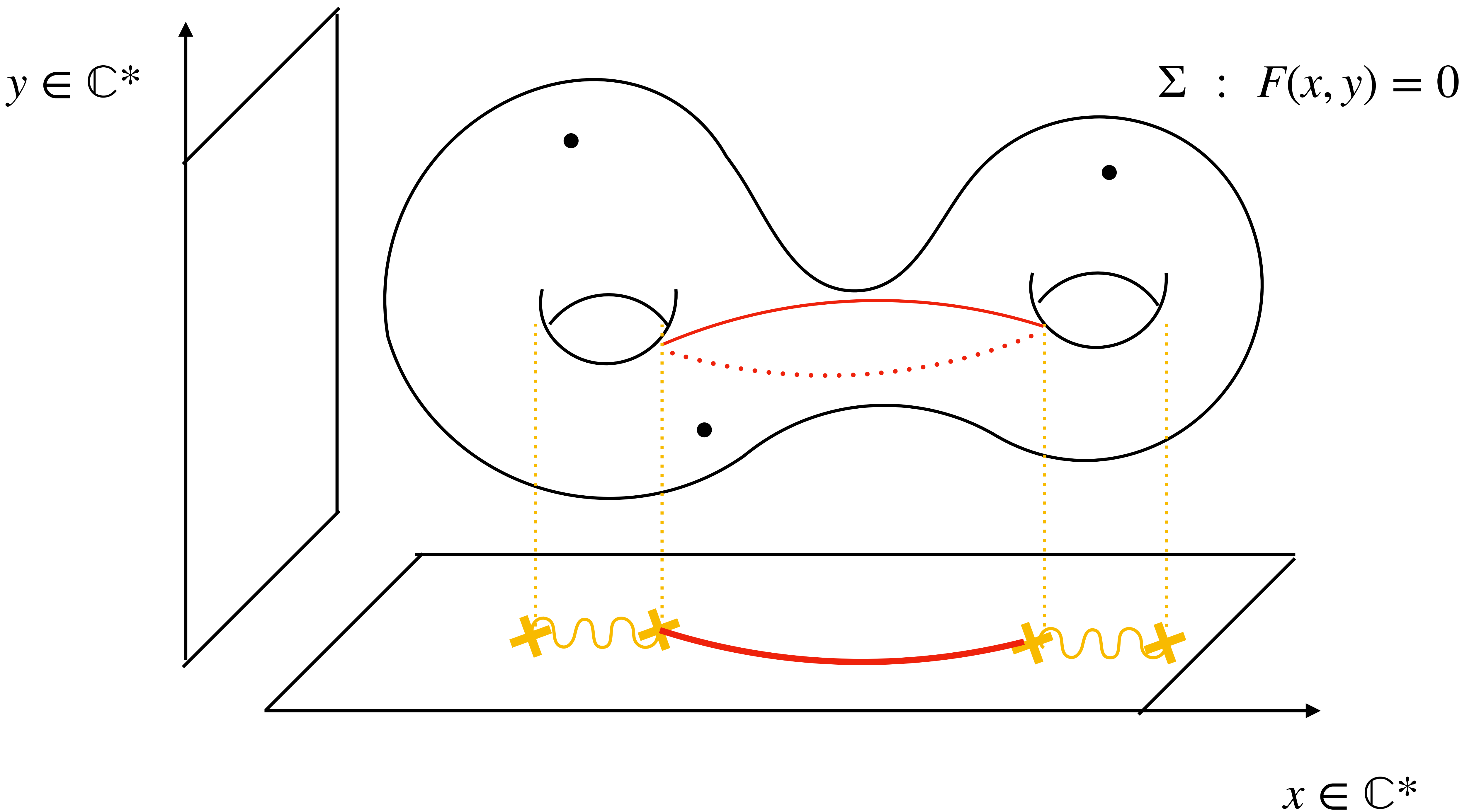

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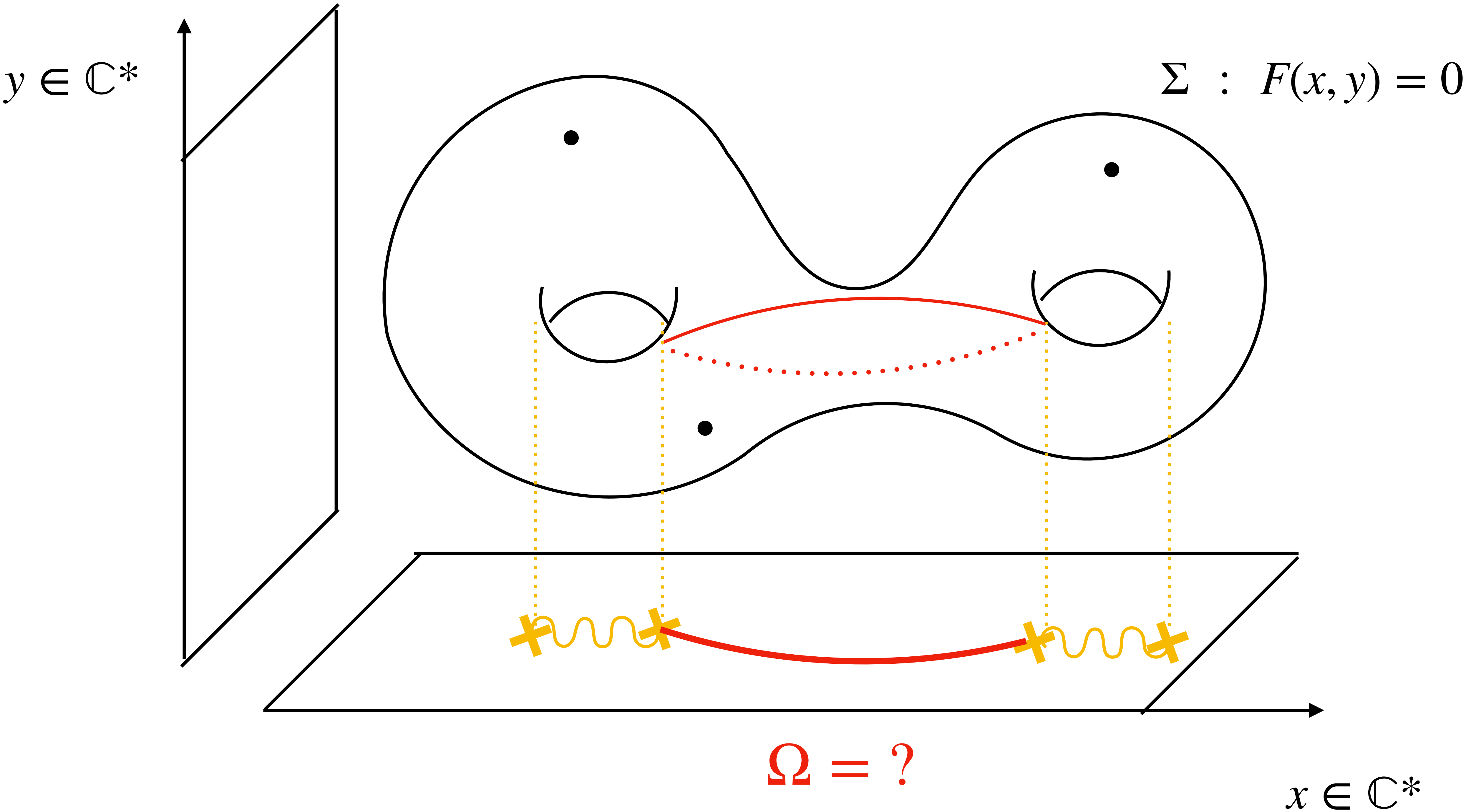

To compute Ω, we take advantage of a phenomenon observed by [Gaiotto-Moore-Neitzke '11], based on the QFT picture.

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- X **(toric) CY 3-fold**

Geometric engineering (QFT)

- **5d gauge theory $T[X]$**

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M theory on $\mathbb{R}^4 \times S^1 \times X$

- X (toric) CY 3-fold
- Introduce M5 on $\mathbb{R}^2 \times S^1 \times L$

BPS States

- M2 brane on $S^1 \times C_2$
- M5 brane on $\mathbb{R} \times S^1 \times C_4$
- New M2-M5 configurations

Geometric engineering (QFT)

- 5d gauge theory $T[X]$
- Obtain a 3d theory $T[L]$ on $\mathbb{R}^2 \times S^1$

BPS States

- El. particles S^1, $M = Vol(C_2)$
- Mag. string $\mathbb{R} \times S^1$, $M = Vol(C_4)$
- 3d BPS states, 3d-5d sector

How does introducing L help compute $\Omega(\gamma)$ for the 5d BPS spectrum?

Think of $T[L]$ as a probe for $T[X]$:

- **Through 3d-5d interactions, the 3d spectrum contains information about 5d BPS states**
- **But the 3d BPS spectrum is easier to compute**

A purpose of Exponential networks is to compute the 3d spectrum *and* extract information about 5d BPS states $\rightarrow \Omega(\gamma)$

Exponential Networks

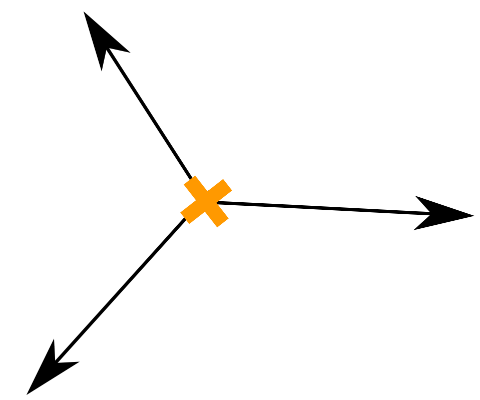
The exponential network $W(\vartheta)$ is a web of trajectories on the x-plane, determined by Σ and $\vartheta \in S^1$ [Eager-Selmani-Walcher'16]

$$(\log y_i - \log y_j + 2\pi i\, n)\, \frac{d \log x}{d\tau} \in e^{i\vartheta}\, \mathbb{R}_+$$

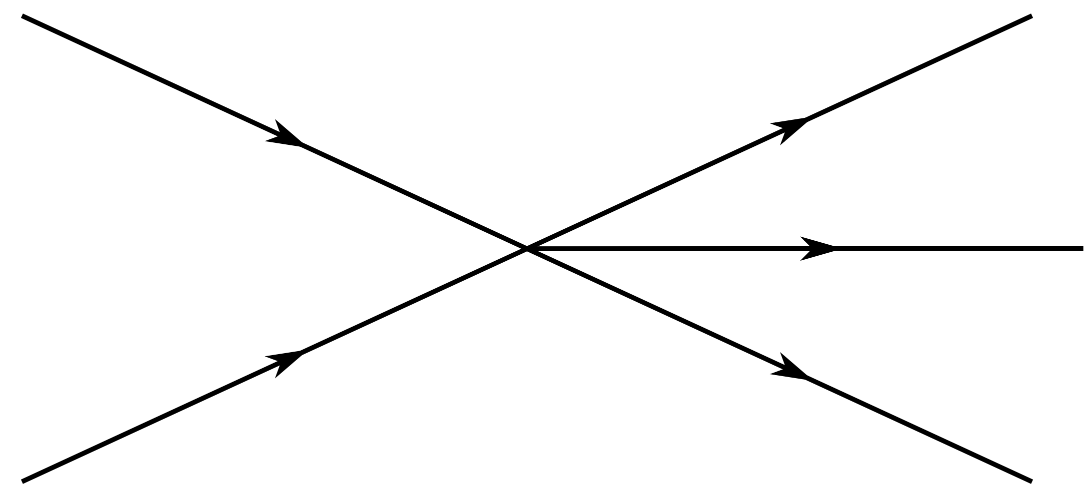
Each trajectory carries combinatorial data, corresponding to 3d BPS states. Determined by topology of $W(\vartheta)$ [Banerjee-L-Romo'18]

Building blocks of $W(\vartheta)$

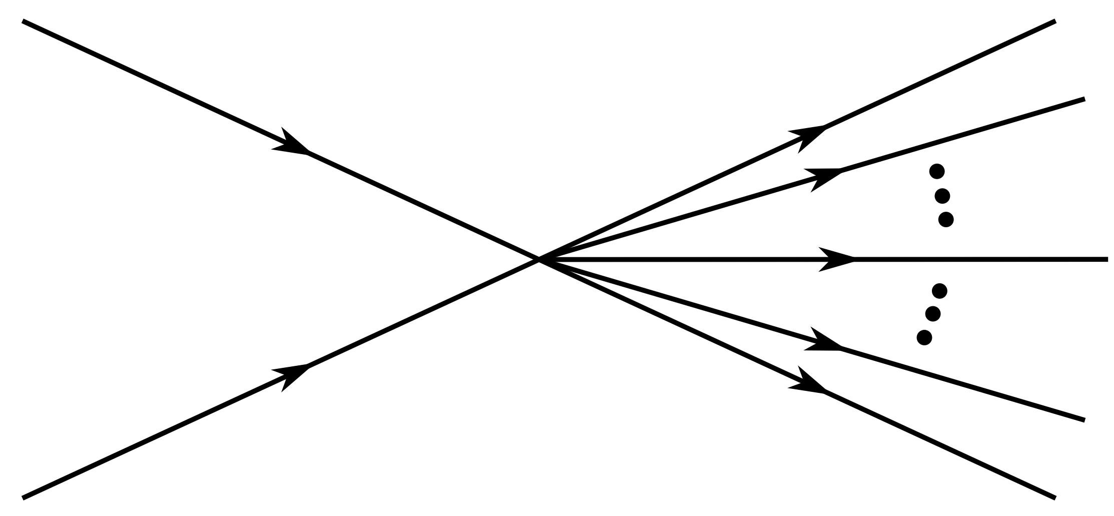
1. Trajectories start from branch points, in triplets



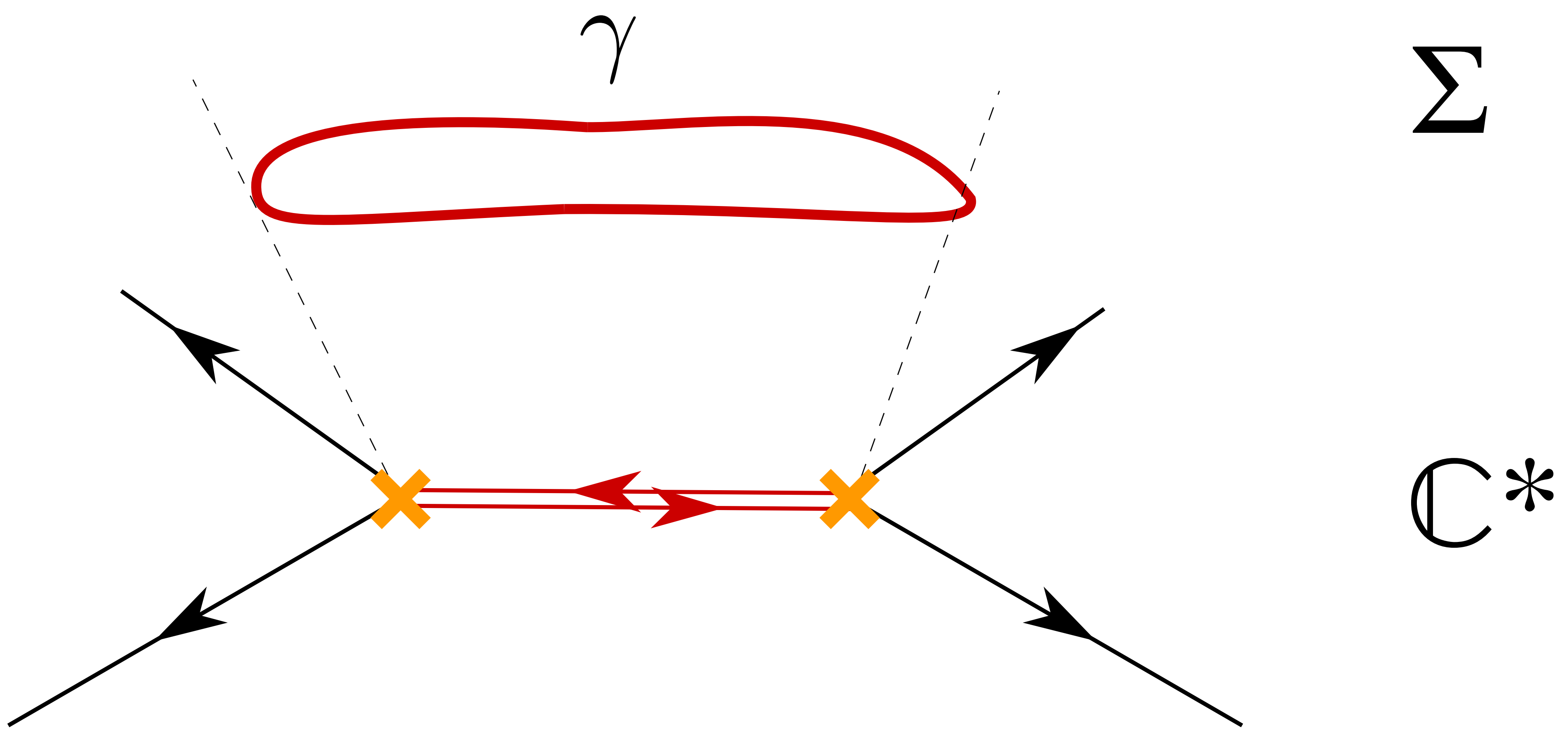
2. As they evolve, intersections may occur



3. New trajectories are born, according to specific rules

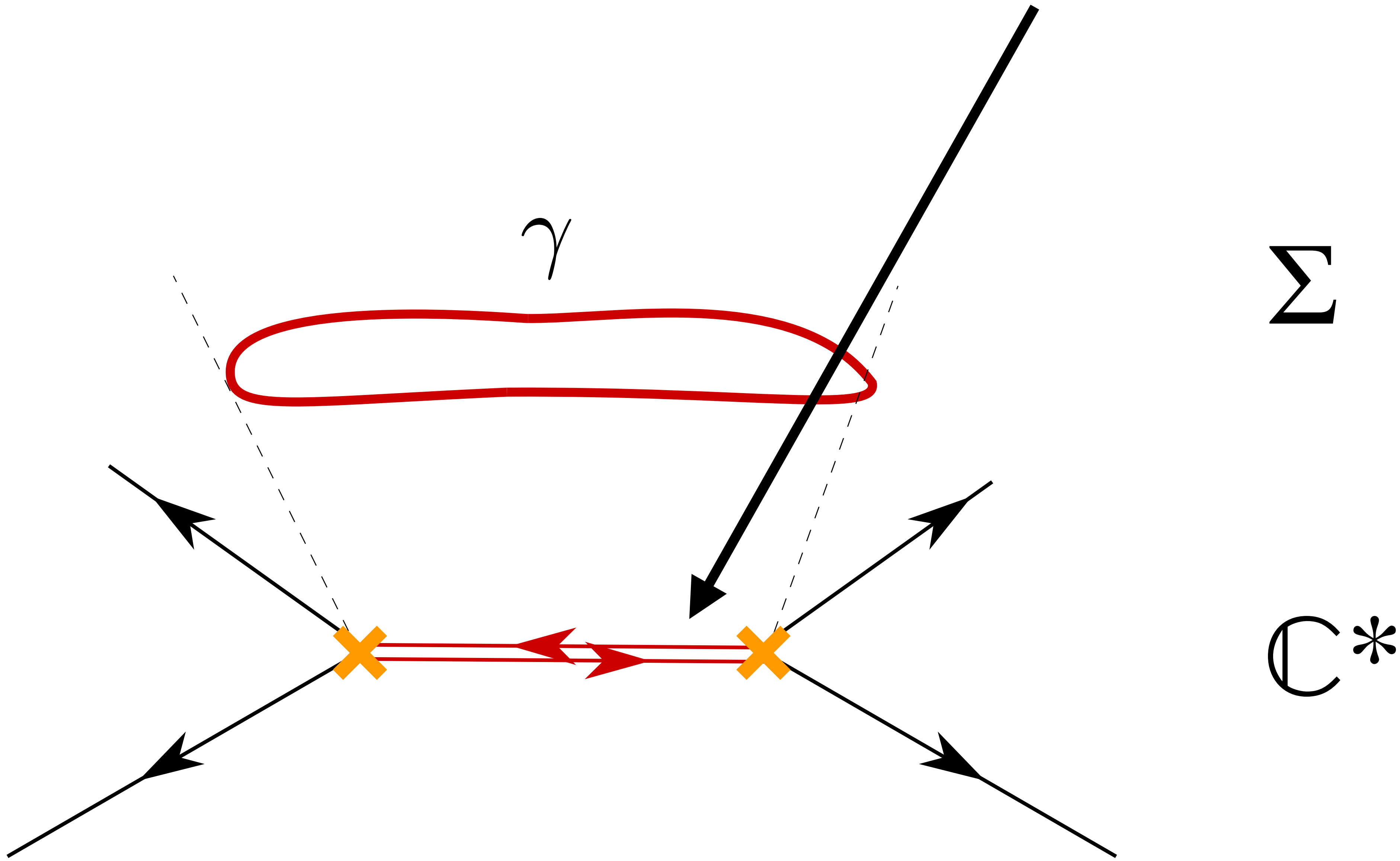


5d BPS states appear as saddle connections at special values of ϑ



5d BPS states appear as saddle connections at special values of ϑ

We can compute $\Omega(\gamma)$ from the topology of the saddle.



Systematic approach to BPS counting

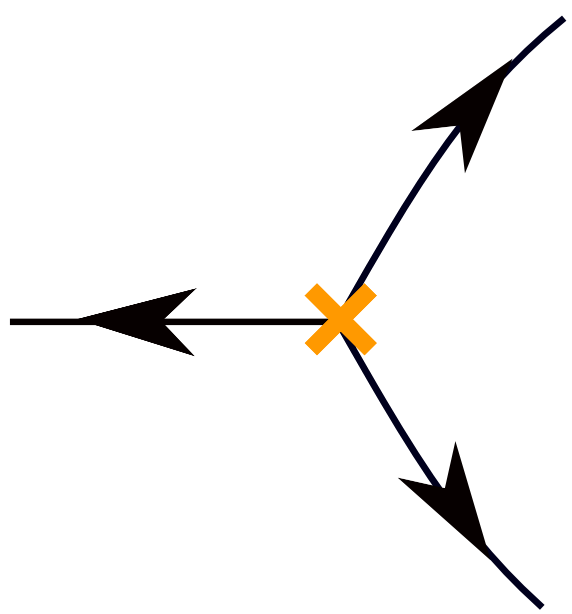
1. **Choose** (X, L) **, identify mirror curve** $F(x, y) = 0$
2. **Plot** $W(\vartheta)$ **, for** $\vartheta \in (0,\pi]$
3. **Identify saddles**
4. **Compute** $\Omega(\gamma)$ **for each saddle**

[Banerjee-L-Romo'18]

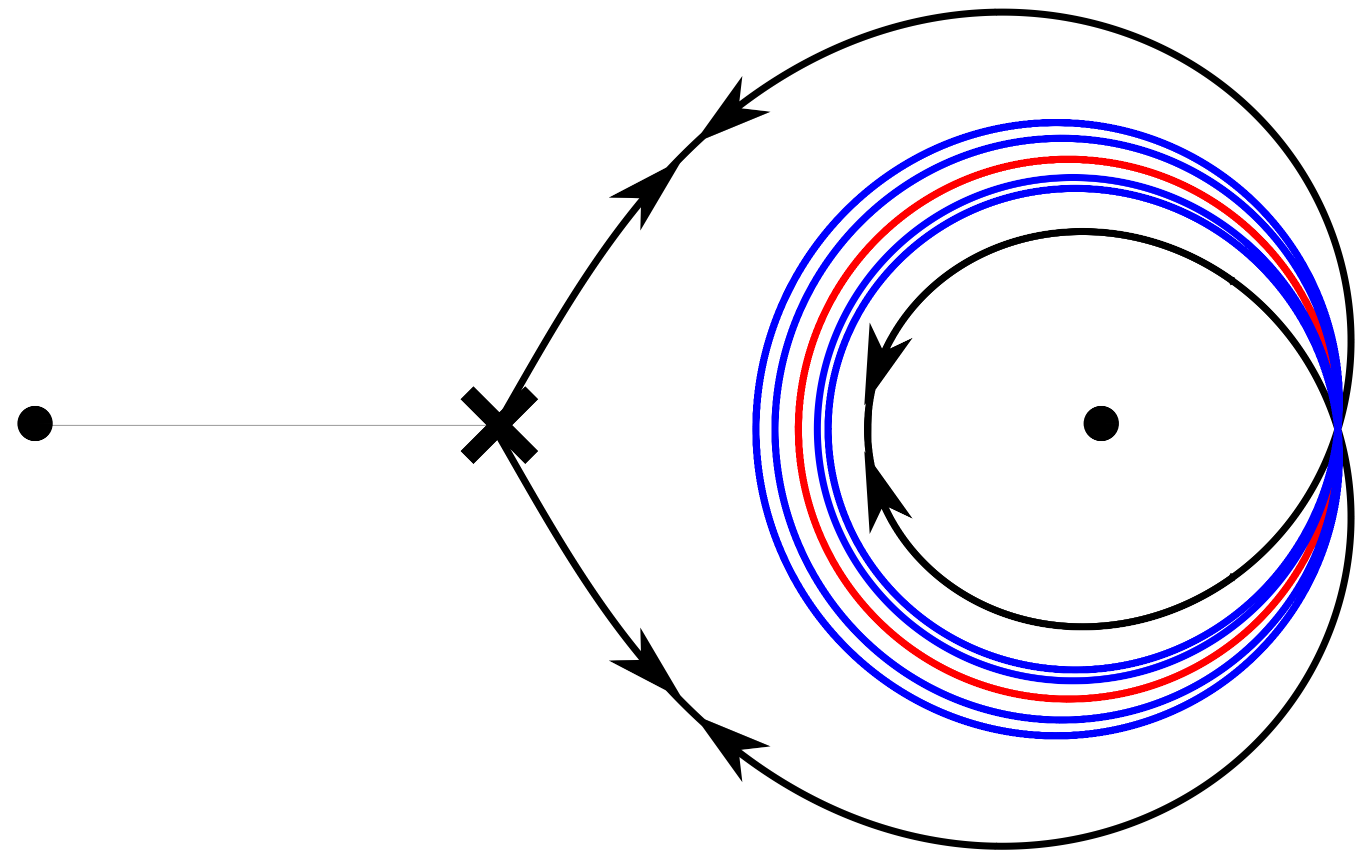
Example 1: $X = \mathbb{C}^3$ **[Banerjee-L-Romo'18]**

$$F(x, y) = 1 + y + xy^2$$

The two sheets $y_{\pm}(x)$ **meet at a single branch point on the** $\mathbb{C}^*$ x**-plane.** $W(\vartheta)$ **starts there:**



There is a tower of saddles at $\vartheta = 0$



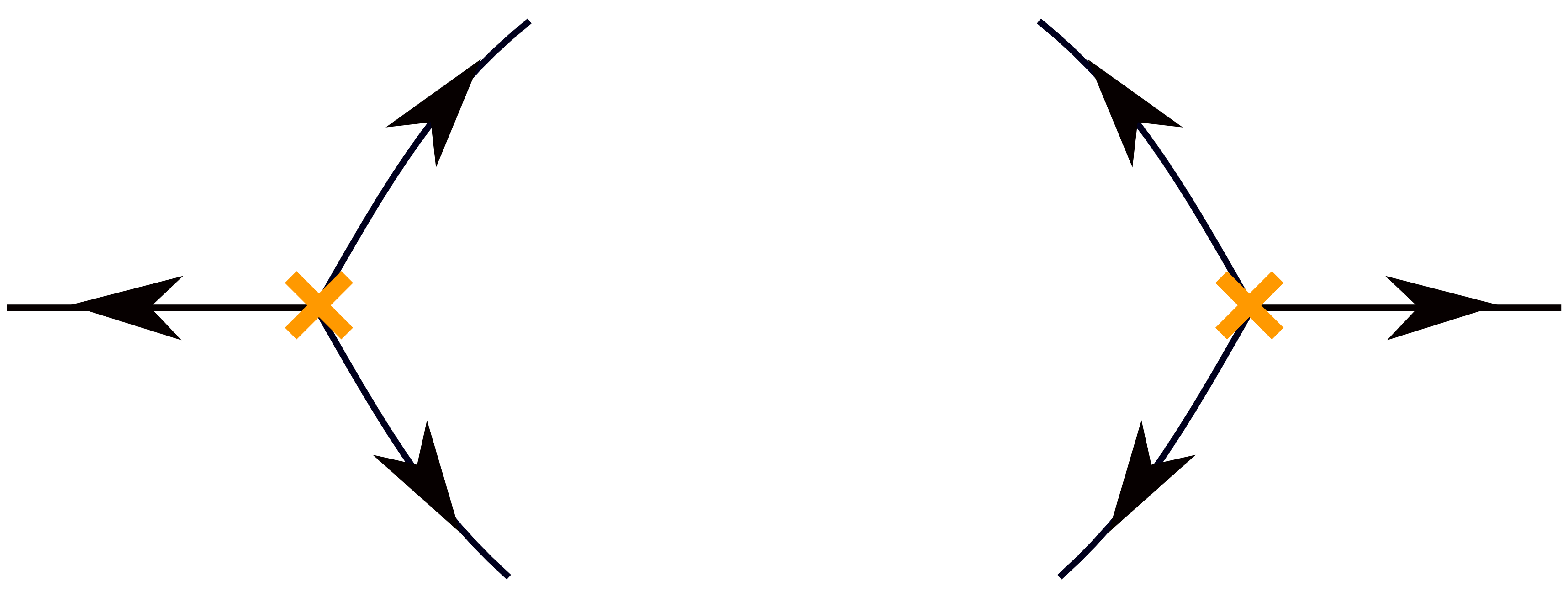
Charges : $k\,\gamma \quad k \geq 1$ **(**$k = 1$ **for the black saddle)**

Masses : $M_k = \dfrac{2\pi}{R} k$ **Degeneracies :** $\Omega(k\gamma) = -1$

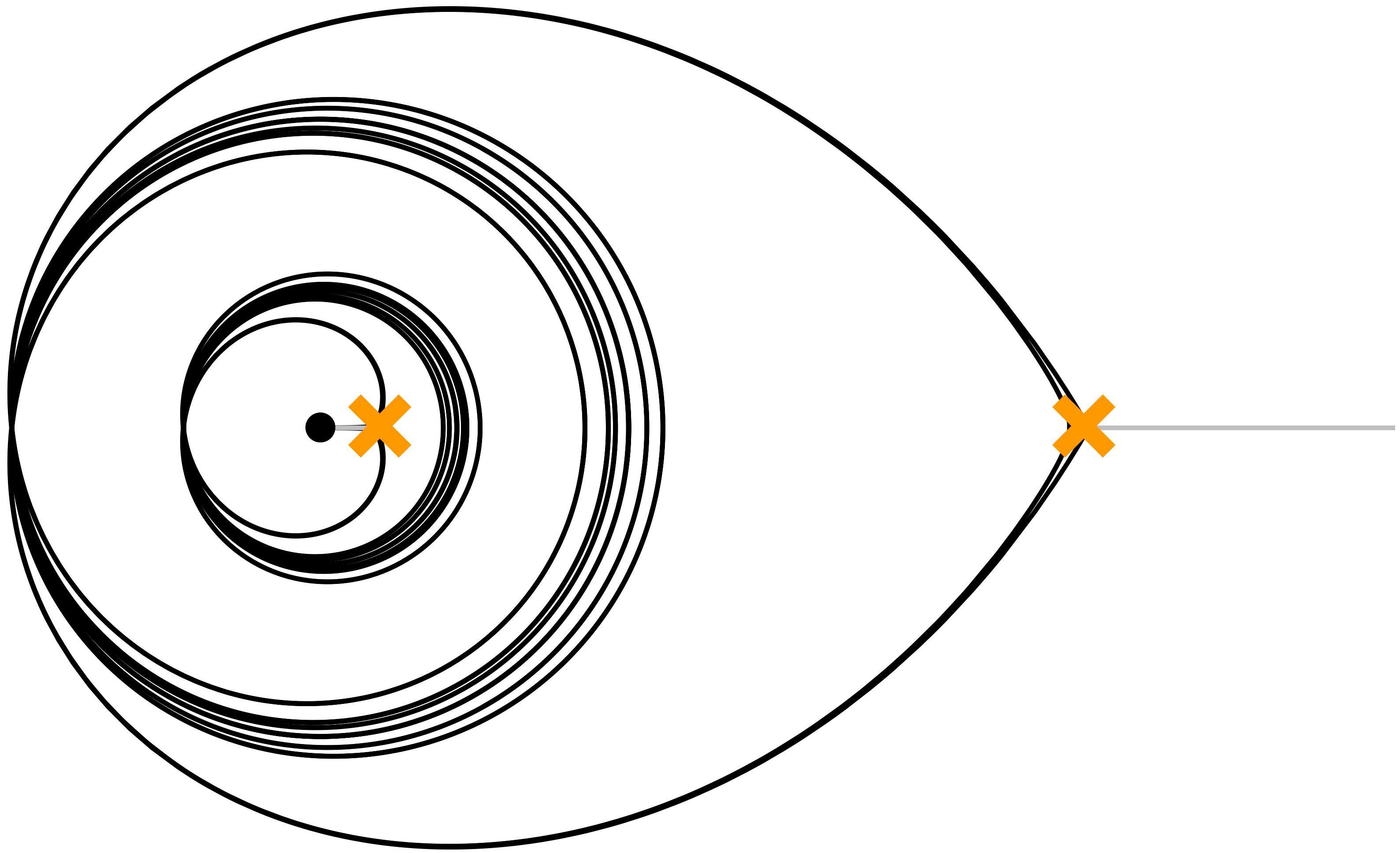
Example 2: $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ [Banerjee-L-Romo'19]

$$F(x, y) = 1 + y + xy + Q\,xy^2$$

Fix $Q > 1$ real-valued. There are now two branch points, with Q-dependent positions. $W(\vartheta)$ **starts there:**



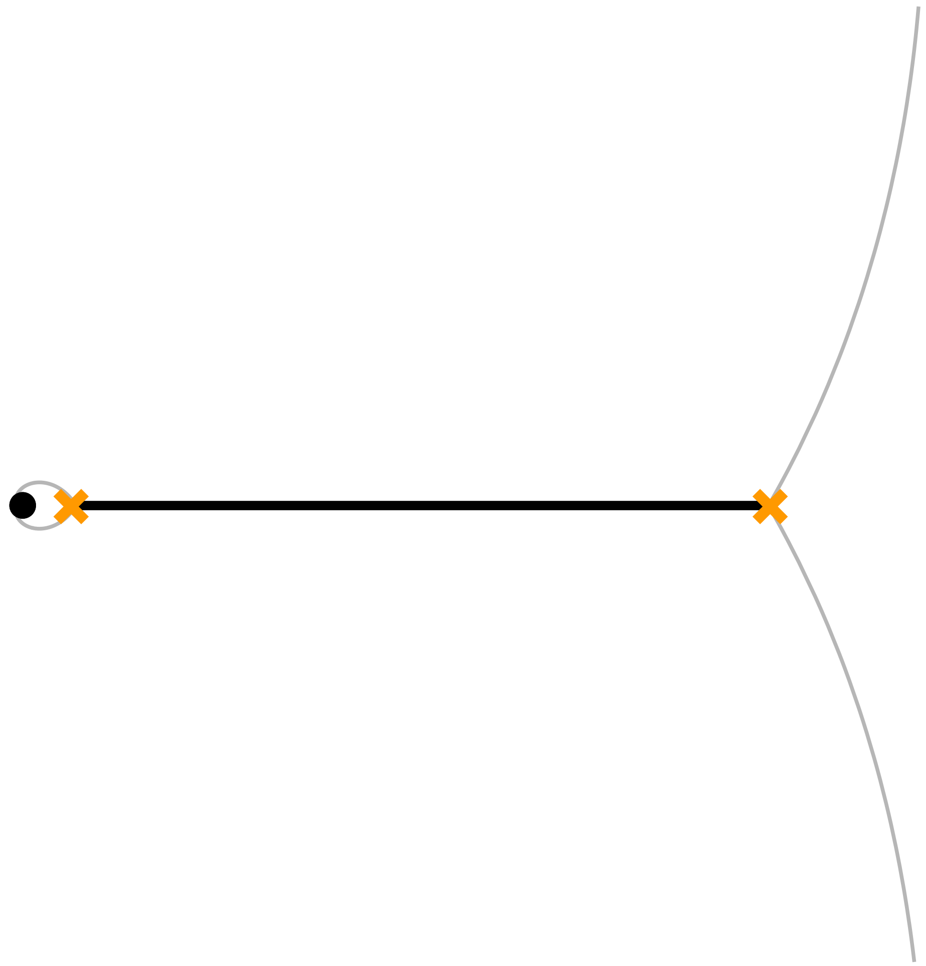
$\vartheta = 0$ **: Two towers of saddles,**



Charges : $k\,\gamma \quad k \geq 1$ $\qquad \gamma =$ **D0**

Masses : $M_k = \dfrac{2\pi}{R} k$ $\qquad$ **Degeneracies :** $\Omega(k\gamma) = -\,2$

$\vartheta = \pi/2$ **: a simple saddle**

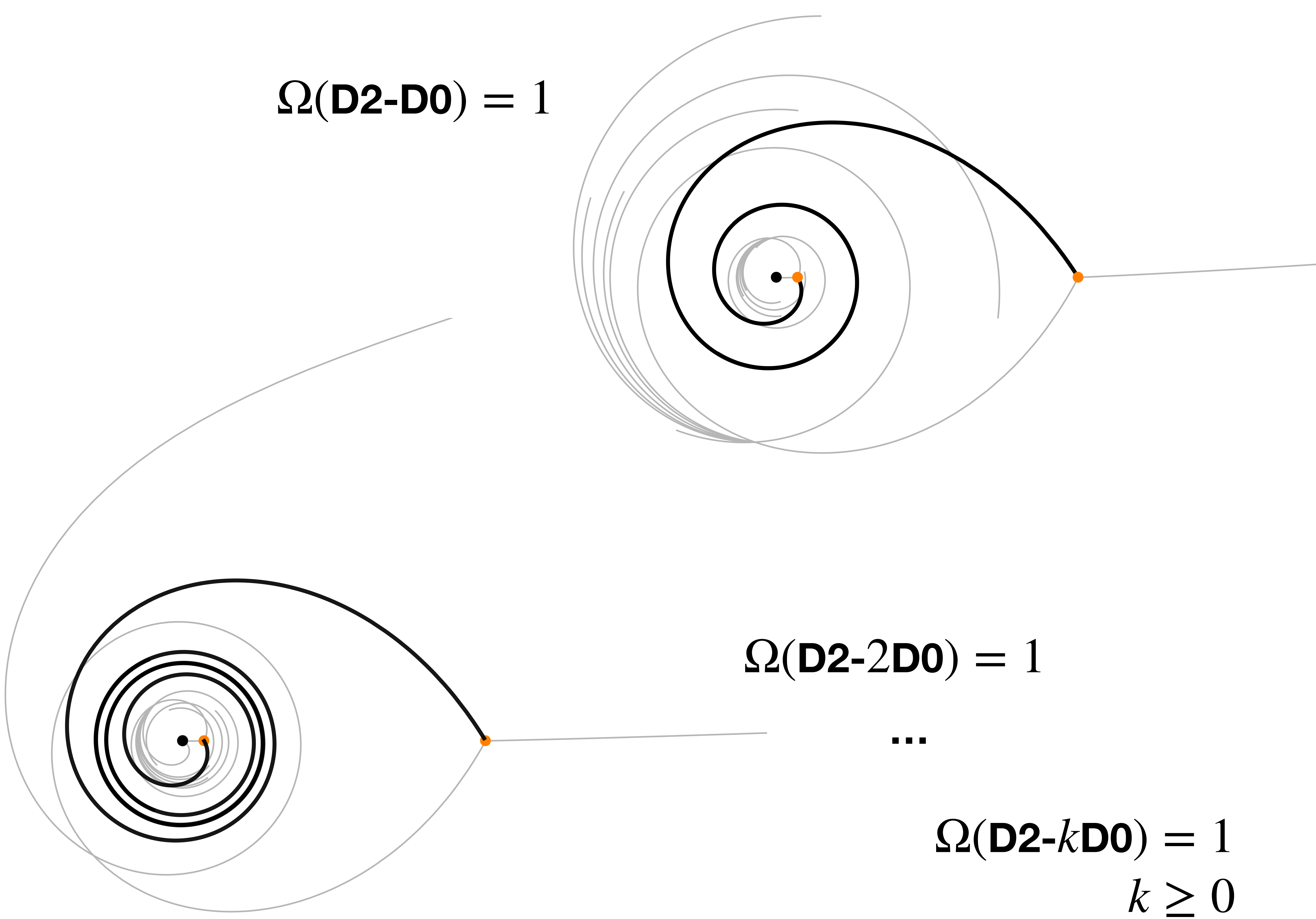


Charge : $\gamma =$ **D2**

Mass : $M = \dfrac{1}{R} log Q$

Degeneracies : $\Omega(\gamma) = 1$,

$\Omega(k\gamma) = 0\,, \quad k > 1$



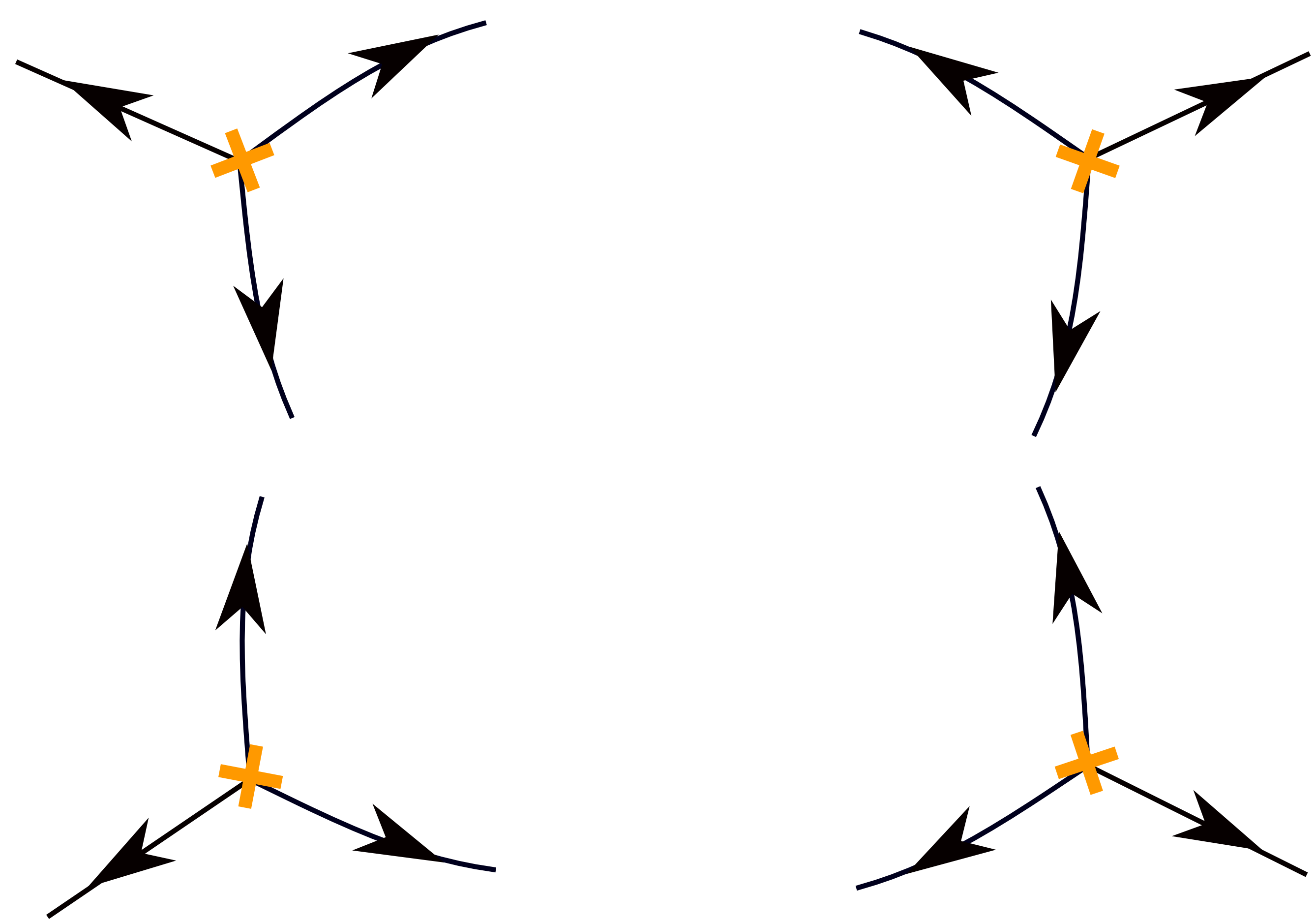
$\Omega(\textbf{D2-D0}) = 1$
$\Omega(\textbf{D2-}2\textbf{D0}) = 1$
...
$\Omega(\textbf{D2-}k\textbf{D0}) = 1$
$k \geq 0$

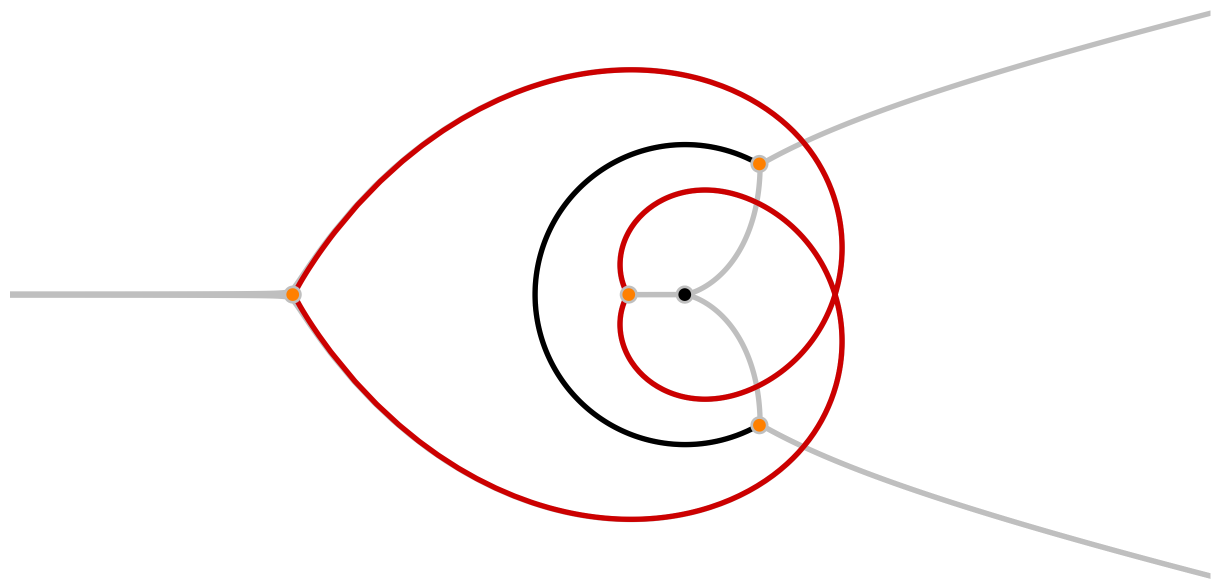
Example 3: $X = O(K) \to \mathbb{F}_0$ **[Banerjee-L-Romo (in prog.)]**

$$F(x, y) = 1 + Q_f\,(y + y^{-1}) + Q_b\,(x + x^{-1})$$

Spectrurm depends on (Q_b, Q_f)**. Fix** $Q_b = -Q_f = -1$

There are now four branch points. $W(\vartheta)$ **starts there:**



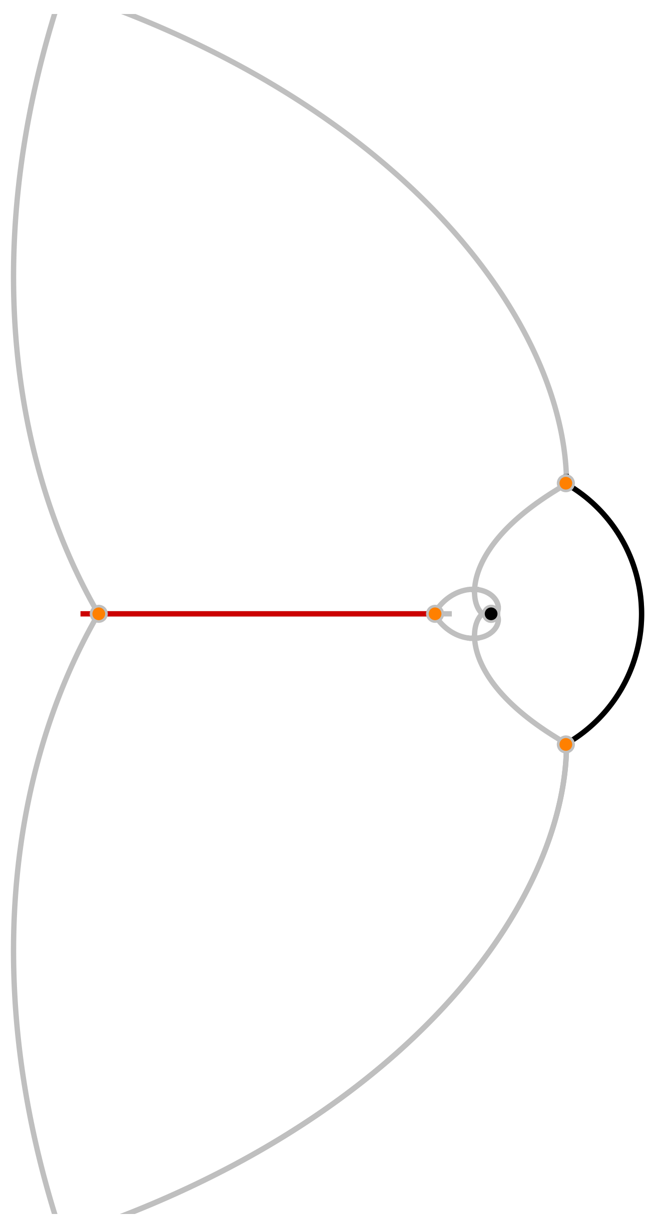


$$\vartheta = 0$$

$$\Omega(\textbf{D0-}\overline{\textbf{D2}}_b\textbf{-D2}_f\textbf{-}\overline{\textbf{D4}}) = 1$$

$$\Omega(\textbf{D0-D2}_b\textbf{-}\overline{\textbf{D2}}_f\textbf{-}\overline{\textbf{D4}}) = 1$$

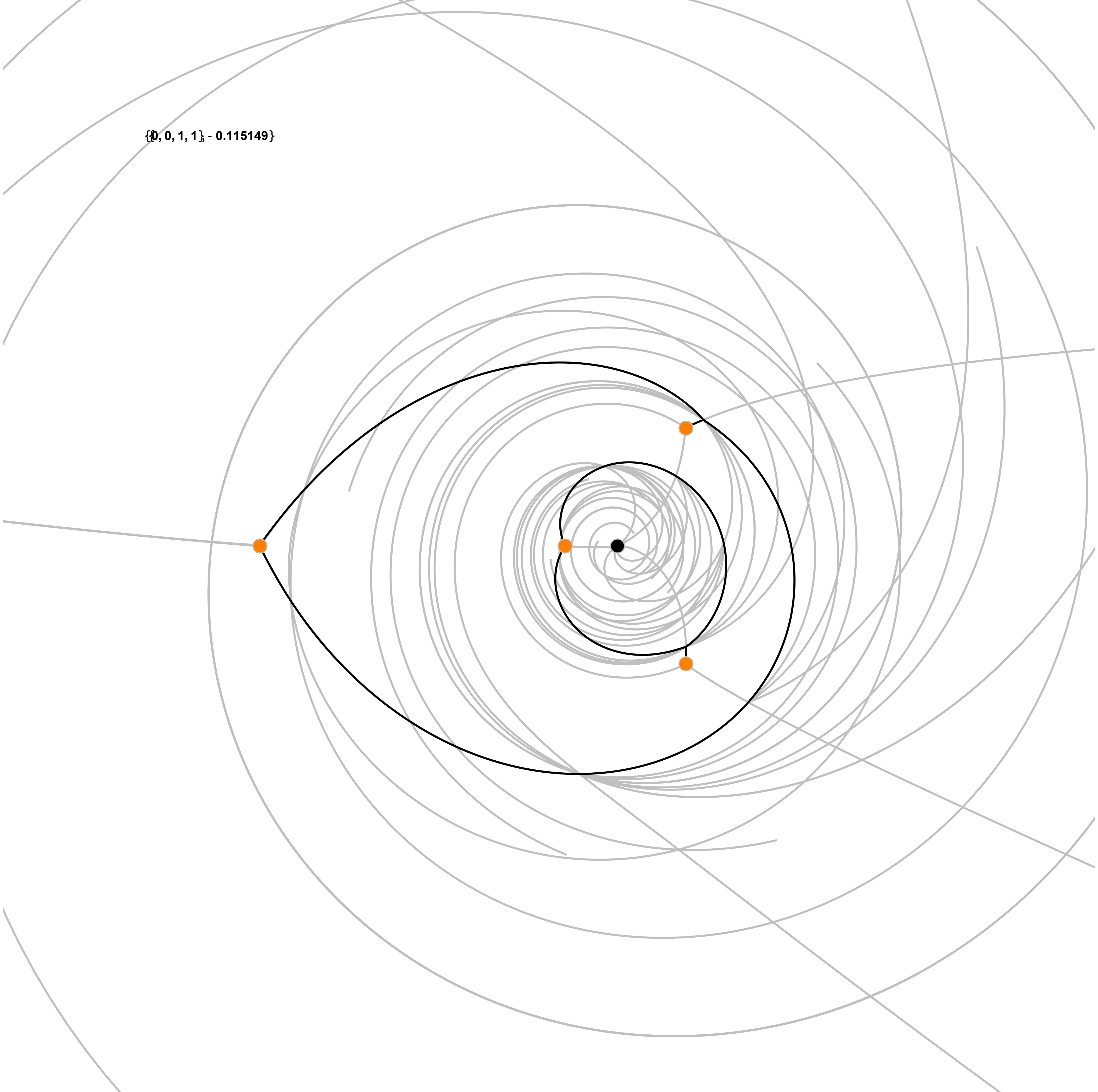
$$\Omega(\textbf{D4}) = 1$$

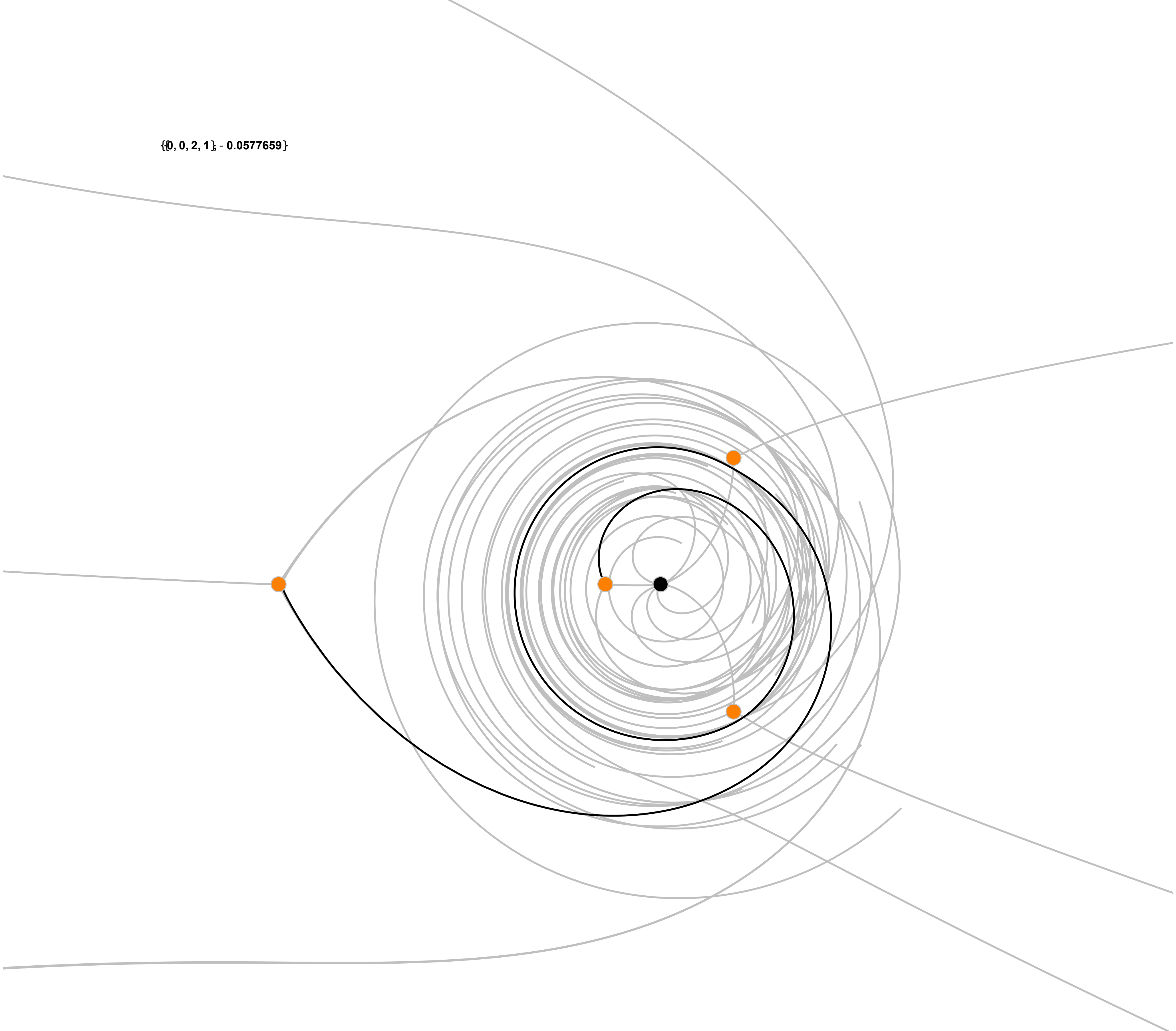


$$\vartheta = \pi/2$$

$$\Omega(\textbf{D2}_b\textbf{-}\overline{\textbf{D4}}) = 1$$

$$\Omega(\overline{\textbf{D2}}_f\textbf{-D4}) = 1$$


{{0, 0, 1, 1}, - 0.115149}


{{0, 0, 2, 1}, - 0.0577659}

Very rich spectrum, including:

D-brane charge	$\Omega(\gamma, u)$
$D4$	1
$D2_f$-$\overline{D4}$	1
$D0$-$D2_b$-$\overline{D2}_f$-$\overline{D4}$	1
$\overline{D2}_b$-$D4$	1
$D0$-$D2_b$-$2\overline{D4}$	-2
$(n+1)D0$-$(n+1)D2_b$-$\overline{D2}_f$-$(2n+1)\overline{D4}$	1
$nD0$-$nD2_b$-$D2_f$-$(2n+1)\overline{D4}$	1
$D0$-$\overline{D2}_f$	-2
$nD0$-$\overline{D2}_b$-$n\overline{D2}_f$-$D4$	1
$(n+1)D0$-$D2_b$-$(n+1)\overline{D2}_f$-$\overline{D4}$	1

As well as infinitely many states with $|\Omega| > 2$ **(wild BPS states)**

Outlook

- **We developed a framework to compute the BPS spectrum of any toric CY 3-fold**
  - **At any (regular) point in moduli space**
  - **Fully systematic**

- **Further applications**
  - **4d limit ($R \to 0$) recover *spectral networks***
    - **Exact WKB analysis**
    - **Hitchin systems**
  - **BPS quivers [Eager-Selmani-Walcher'16]**
  - **Framed BPS states**
  - **More choices of L, e.g. augmentation curves**