Quantization of Poisson Hopf algebras

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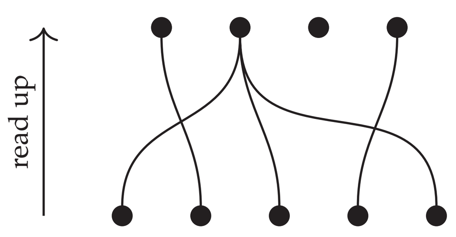
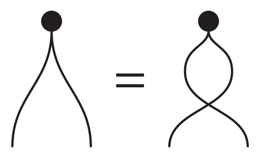
Commutative Hopf algebras

Symmetric monodial category Com: (i.e. finite sets)



modulo



Let $\mathcal{C}$ be any symmetric monoidal category.

Theorem 1. *The category of symmetric lax monoidal functors $N\colon \mathsf{Com} \to \mathcal{C}$ satisfying the nerve condition**

$$\Longleftrightarrow$$

The category of commutative Hopf algebras in $\mathcal{C}$.

*nerve condition:

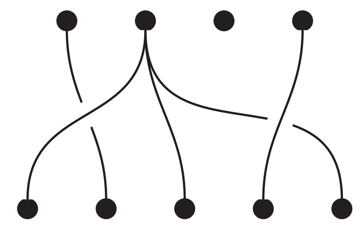
$$N(2)^{\otimes n} \xrightarrow{\text{lax functor}} N(2n) \xrightarrow{N(\cdots)} N(n+1), \qquad N(2)^{\otimes n} \xrightarrow{\cong} N(n+1)$$

$$\text{and} \quad 1_{\mathcal{C}} \xrightarrow{\cong} N(0) \xrightarrow{\cong} N(1)$$

Explanation. N is (functions on) the symmetric nerve of a group G, i.e. $H = \mathcal{O}(G) = N(2)$.

Hopf algebras with invertible antipode

Braided monodial category BrCom:



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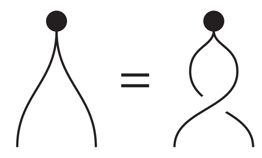
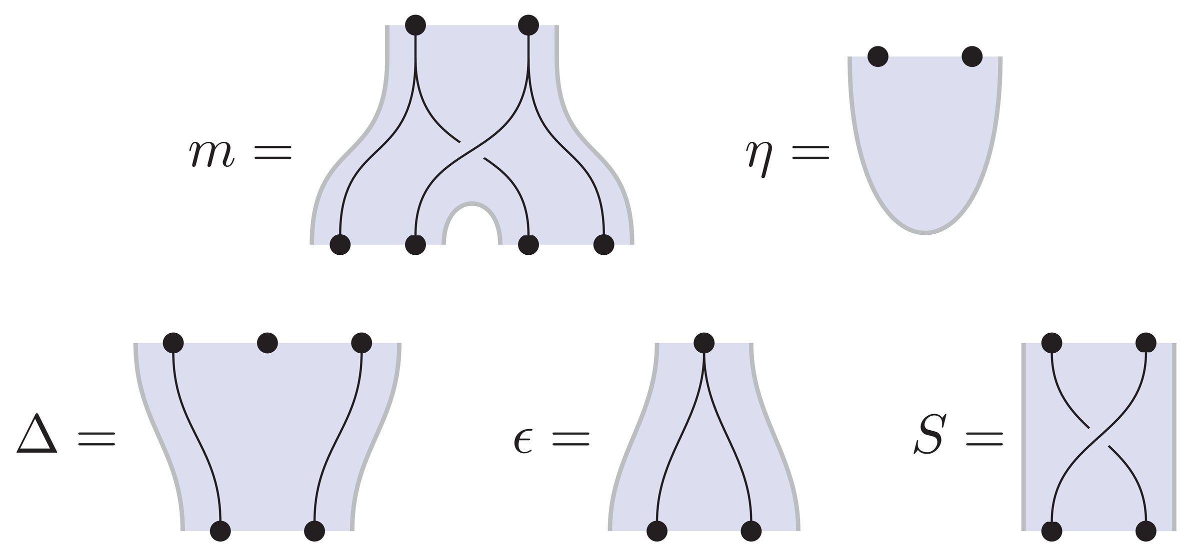


Let $\mathcal{C}$ be any braided monoidal category.

Theorem 2. *The category of braided lax monoidal functors $N\colon \mathsf{BrCom} \to \mathcal{C}$ satisfying the nerve condition**

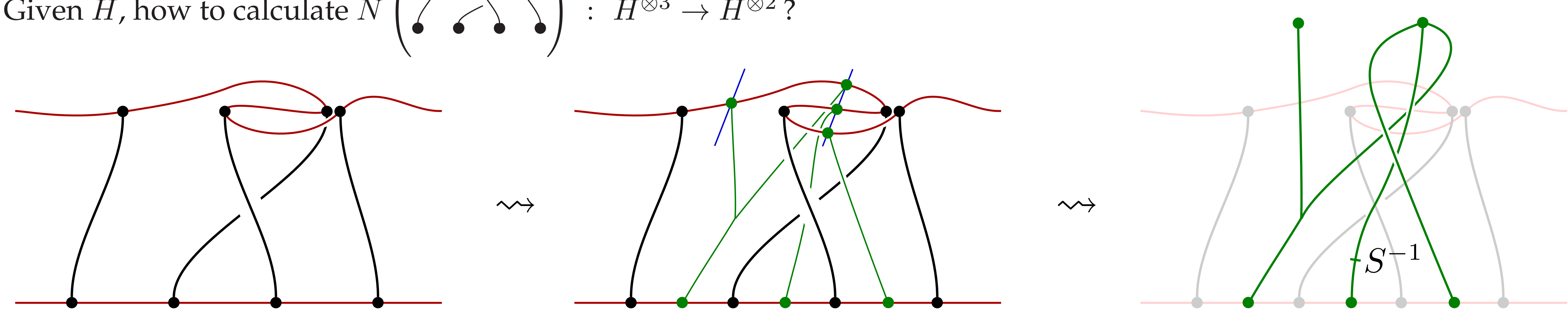
$$\Longleftrightarrow$$

The category of braided Hopf algebras with an invertible antipode in $\mathcal{C}$.



The nerve functor N

Example. Given H, how to calculate $N\left(\cdots\right) : H^{\otimes 3} \to H^{\otimes 2}$?

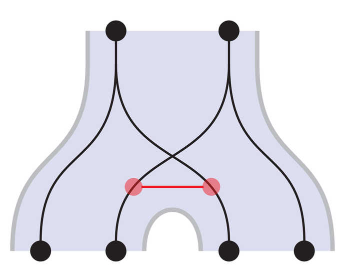


Quantization

Infinitesimally braided monodial category iCom:
(linear combinations of) Com + horizontal chords

Let $\mathcal{C}$ be any infinitesimally braided monoidal category.

Theorem 3. *The category of infinitesimally braided lax monoidal functors $N\colon \mathsf{iCom} \to \mathcal{C}$ satisfying the nerve condition* $\Longleftrightarrow$ The category of Poisson Hopf algebras in $\mathcal{C}$.*



Poisson bracket

Choose Φ a *rational Drinfeld associator*.

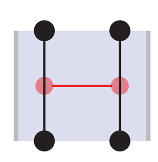
Quantization of N is the composition

$$\mathsf{BrCom} \xrightarrow{\Phi} \mathsf{iCom}^{\Phi}_{\hbar} \xrightarrow{N_\hbar} \mathcal{C}^{\Phi}_{\hbar}.$$

It is universal: prop map $\mathsf{Hopf} \to \mathsf{PoissHopf}_\hbar(\mathbb{Q})$

Lie bialgebras

- $(\mathfrak{g}, [\,,\,], \delta)$ a Lie bialgebra
- $U\mathfrak{g}$ is a commutative Hopf algebra in $\mathrm{Vect}^{\mathrm{op}}$
- δ extends to a Poisson bracket on $U\mathfrak{g} \in \mathrm{Vect}^{\mathrm{op}}$

 $= -\frac{1}{2}[\,,\,] \circ \delta$

- Quantization: Hopf algebra in $\mathrm{Vect}_\hbar$

Future Directions

- Any higher group(oid) has a nerve $\mathsf{Com} \to \mathcal{C}$. What are the semi-classical and quantum versions?
  (works for suitable Hopf algebroids)
- Chord diagrams P_n form a functor $\mathsf{BrCom} \to \mathrm{Vect}$ by
  $$n \mapsto P_{n-1}.$$

References

[1] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$
[2] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I.
[3] J. Pulmann, P. Ševera, Quantization of Poisson Hopf algebras. `Arxiv:1906.10616`