# Quantum Hall Effect

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# Transport in condensed matter

- Endless source of beautiful mathematical problems, combining different mathematical methods (analysis, probability, geometry...)

- Loosely speaking: understand how an electron gas, formed by a large number of particles, responds to external perturbations (e.g.: external electric field, variation of chemical potential, etc).

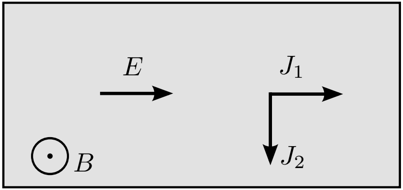
- The motion of the individual charge carriers is described by the Schrödinger equation. For one particle with wave function $\psi \in L^2(\mathbb{R}^d)$:

$$i\partial_t \psi_t = H\psi_t \,, \quad H = H^* = \text{Hamiltonian}, \quad \psi_0 = \psi \,.$$

We will be interested in systems formed by $\infty$-many particles. In some cases, quantum mechanical effects remain visible at a macroscopic scale.

# The Hall effect - Edwin Hall 1879

- Setting: Ultrathin materials exposed to transverse magnetic field $B$ and a weak in-plane electric field $E$.



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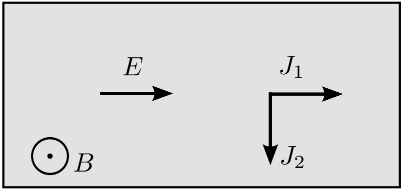



- Linear response (weak $E$):

$$J_1 = \sigma_{11} E \, , \qquad J_2 = \sigma_{21} E$$

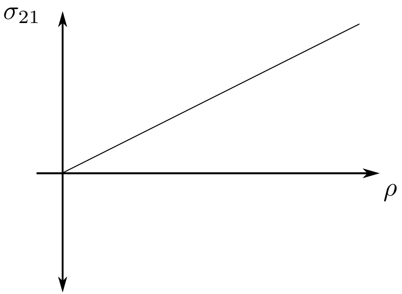
$\sigma_{11}$ = longitudinal conductivity, $\qquad \sigma_{21} = -\sigma_{12}$ = Hall conductivity.

From the laws of classical electrodynamics:

$$\sigma_{21} = \frac{e^2}{h}\nu \, , \qquad \nu = \frac{\rho}{|e|\frac{B}{hc}} \qquad (\rho = \text{density of charge carriers})$$

# The Hall effect

- Classical prediction: linear behavior of transverse conductivity



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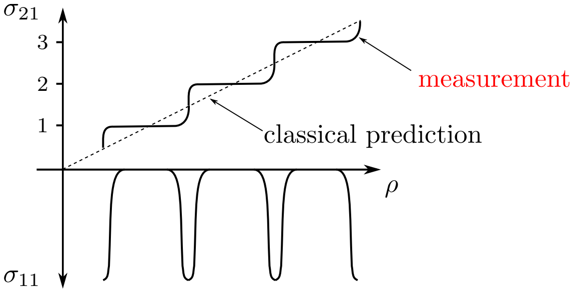
- von Klitzing '80. Experiment on GaAs-heterostructures (insulators). Large magnetic field $B$ and small temperature $T$.



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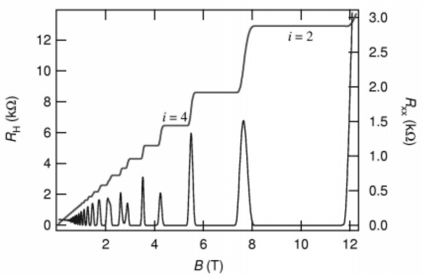

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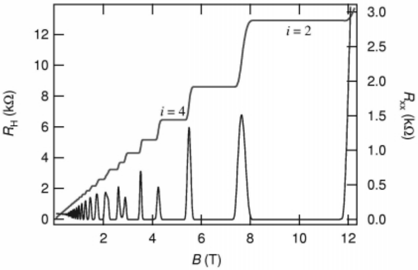


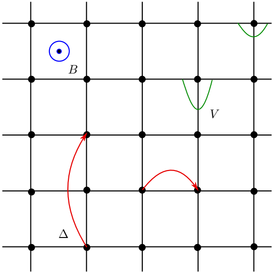

- Integer Quantum Hall effect: $\sigma_{21}$ is quantized, with $10^{-9}$ precision!

$$\sigma_{21} = \frac{e^2}{h} n \,, \qquad n \in \mathbb{Z} \,.$$

Purely quantum phenomenon. First example of topological insulator.

# Theory: setting

- Electron gas on an infinite $2d$ lattice as a model for a crystal, in the tight-binding approximation. E.g.: $\mathbb{Z}^2$.




where:

- $\Delta$ : lattice hopping (e.g.: lattice Laplacian);
- $V$: external potential (periodic potential, impurities...)
- $B$: constant magnetic field.

# Theory: setting

- Example of single-particle Hamiltonian, on $\ell^2(\mathbb{Z}^2)$:

$$H = -\Delta_A + V \,, \qquad (V\psi)(x) = V(x)\psi(x)$$

where $A$ is a vector potential generating the magnetic field:

$$\Delta_A(x;y) = \Delta(x;y)e^{i\int_{x\to y} d\ell\cdot A(\ell)} \,, \qquad \int_{\partial(\text{plaquette})} d\ell \cdot A(\ell) = \text{Flux}(B)$$

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- Ground state of many, noninteracting fermions: "fill the Fermi sea", up to the Fermi energy $\mu$. Averages of observables, $O = O^*$:

$$\langle O \rangle_\mu := \text{Tr}_{\ell^2(\mathbb{Z}^2)}\, O P_\mu \;, \qquad P_\mu = \chi(H \leq \mu) = \text{Fermi projector.}$$

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- Important assumption: insulating behavior,

$$|\langle \delta_x, P_\mu \delta_y \rangle| \leq C e^{-c|x-y|} \ .$$

True if $\mu \notin \sigma(H)$ (spectral gap), or if $\mu \in$ mobility gap (later).

# Linear response

- Time-dependent perturbation, for $t \le 0$ ($\eta \ge 0$ small):

$$A(\ell) \to A(\ell) + a(\eta t) \,, \qquad \partial_t a(\eta t) =: E(t) \,, \qquad a(-\infty) = 0 \,,$$

nontrivial time evolution:

$$i\partial_t P(t) = [H(t)\,, P(t)] \,, \qquad P(-\infty) = P_\mu \,.$$

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- The final goal is to understand the $E$-dependence of:

$$\mathcal{J} = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \operatorname{Tr} \chi(x \in \Lambda_L) J P(0)$$

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- Linear response: expand the state in $E$,

$$\mathcal{J} = \sigma E + o(E)\;, \qquad E \equiv E(0)\;,$$

  where $\sigma$ = conductivity matrix, given by Kubo formula:

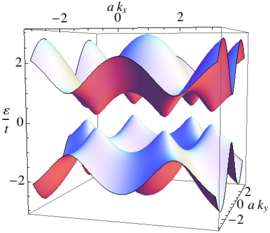
$$\sigma_{ij} = \lim_{L\to\infty} \frac{i}{|\Lambda_L|} \mathrm{Tr}\, \chi(x \in \Lambda_L) P_\mu [[P_\mu, X_i], [P_\mu, X_j]]\;.$$

# Gapped systems

- Let $H$ on $\ell^2(\mathbb{Z}^2;\mathbb{C}^M)$, translation invariant: $H(x;y) \equiv H(x-y)$ .
  Bloch Hamiltonian: $\hat{H}(k)$, with $k \in \mathbb{T}^2$. Eigenvalue equation:

$$\hat{H}(k)\varphi_i(k) = \varepsilon_i(k)\varphi_i(k) \ , \qquad i = 1, \ldots, M,$$

  where $\varepsilon_i(k)$ define the energy bands. Suppose $\mu$ is in a spectral gap.



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- Let $P_\mu = \bigoplus_{k\in\mathbb{T}^2} \hat{P}(k)$ and suppose $\hat{P}(k) = |\varphi_1(k)\rangle\langle\varphi_1(k)|$. Then:

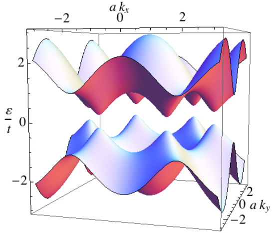
$$\sigma_{12} = \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2} \vec{\nabla} \times \langle \varphi_1(k), i\vec{\nabla}_k \varphi_1(k)\rangle \in \frac{1}{2\pi}\mathbb{Z}$$

- More generally, $\sigma_{12}$ = Chern number of Bloch bundle,

  $\mathcal{E}_{\rm B} = \{(k,u) \in \mathbb{T}^2 \times \mathbb{C}^M \mid u \in \mathrm{Ran}\,\hat{P}(k)\}$ [TKNN; Avron, Seiler, Simon]

# Plateaux

- In the previous example we crucially used that $\mu$ lies in a spectral gap. E.g.: $\varphi(k)$ eigenstate of $\hat{H}(k)$ corresponding to the lowest eigenvalue.




- This setting cannot explain the emergence of plateaux! Varying $\mu$ in the spectral gap does not change the density of charge carriers.

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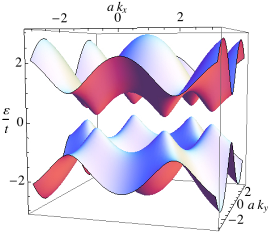
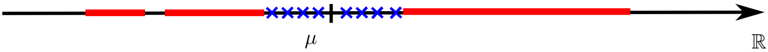



- Strong disorder: $V(x) = \lambda\omega_x$, $|\lambda| \gg 1$, $\{\omega_x\}$ i.i.d. $\Rightarrow$ Anderson loc. For $\mu$ in a mobility gap: $\sigma_{12}$ is integer-valued, and continuous in $\mu$!




[Bellissard, van Elst, Schulz-Baldes; AS$^2$; Aizenman, Graf]

# Conclusion and open questions

- The precise explanation of the IQHE is a major achievement of mathematical physics.
- It pioneered the study of topological insulators and the notion of topological phase of matter.
- Many-body interactions?

  In the last years, the stability of the IQHE against weak many-body interactions has been rigorously understood.
- Two major open questions:
  - Existence of plateaux for many-body quantum systems (many-body localization?)
  - Strong interactions should have a dramatic consequence on transport. Fractional quantum Hall effect:

    $$\sigma_{12} \in \frac{1}{2\pi}\mathbb{Q}\,.$$

    Proof from a microscopic model?