Landau-Pekar equations and quantum fluctuations for the dynamics of a strongly coupled polaron

Nikolai Leopold
(Saffirio group, University of Basel)

Joint work with David Mitrouskas, Simone Rademacher, Benjamin Schlein and Robert Seiringer

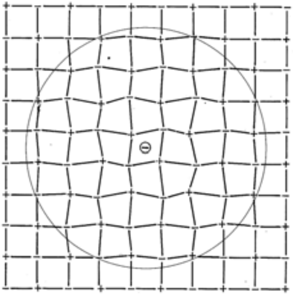
SwissMAP General Meeting
September 9, 2020

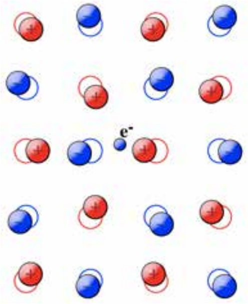
Outline of the talk

- Fröhlich Polaron model
- Landau-Pekar equations
- Main results
- Sketch of the proof

The Polaron

An electron in an ionic crystal polarizes its surroundings.

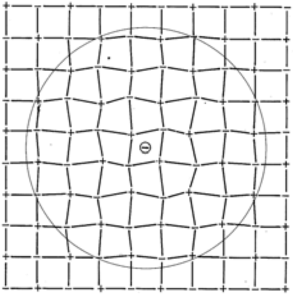


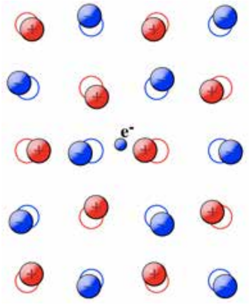


Figures: Madelung Festkörpertheorie II, Wikipedia

The Polaron

An electron in an ionic crystal polarizes its surroundings.



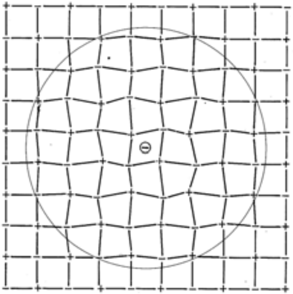


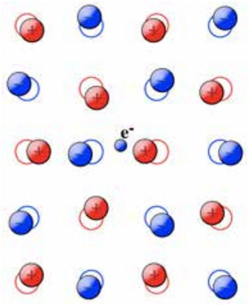
Figures: Madelung Festkörpertheorie II, Wikipedia

- 1937: The Fröhlich Model,

The Polaron

An electron in an ionic crystal polarizes its surroundings.





Figures: Madelung Festkörpertheorie II, Wikipedia

- 1937: The Fröhlich Model,
- 1948: The Landau-Pekar equations.

The Fröhlich Model

$$i\partial_t \Psi_t = H\,\Psi_t,$$

$$H = -\Delta + \sqrt{\alpha}\int d^3k\,\left(G_x(k)a_k^* + \overline{G_x(k)}a_k\right) + \int d^3k\, a_k^* a_k,$$

$$[a_k, a_l^*] = \delta(k-l) \quad \text{and} \quad [a_k, a_l] = [a_k^*, a_l^*] = 0.$$

Remarks:

- $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \left[\oplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}\right]$
- $G_x(k) = |k|^{-1}\, e^{-ik\cdot x} \notin L^2(\mathbb{R}^3)$ but e^{-iHt} can be defined via the associated quadratic form,

The Fröhlich Model

$$\alpha^2 i\partial_t \Psi_t = H\,\Psi_t,$$

$$\alpha^{-2} H = -\Delta + \cancel{\sqrt{\alpha}} \int d^3k\, \left(G_x(k) a_k^* + \overline{G_x(k)} a_k \right) + \int d^3k\, a_k^* a_k,$$

$$[a_k, a_l^*] = \alpha^{-2}\delta(k-l) \quad \text{and} \quad [a_k, a_l] = [a_k^*, a_l^*] = 0.$$

Remarks:

- $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \left[\oplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n} \right]$
- $G_x(k) = |k|^{-1}\, e^{-ik\cdot x} \notin L^2(\mathbb{R}^3)$ but e^{-iHt} can be defined via the associated quadratic form,
- strong coupling units: $x \to \alpha^{-1}x$, $a_k \to \alpha^{-1/2} a_{\alpha^{-1}k}$, $t \to \alpha^2 t$

The Fröhlich Model

$$i\partial_t \Psi_t = H\,\Psi_t,$$

$$H = -\Delta + \int d^3k\, \left(G_x(k)a_k^* + \overline{G_x(k)}a_k\right) + \int d^3k\, a_k^* a_k,$$

$$[a_k, a_l^*] = \alpha^{-2}\delta(k-l) \quad \text{and} \quad [a_k, a_l] = [a_k^*, a_l^*] = 0.$$

Remarks:

- $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \left[\oplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}\right]$
- $G_x(k) = |k|^{-1}\, e^{-ik\cdot x} \notin L^2(\mathbb{R}^3)$ but e^{-iHt} can be defined via the associated quadratic form,
- strong coupling units: $x \to \alpha^{-1}x$, $\alpha_k \to \alpha^{-1/2}a_{\alpha^{-1}k}$, $t \to \alpha^2 t$
- classical behavior of the phonon field for large α.

Pekar product states

Weyl operator: $W(\varphi) = e^{a^*(\varphi) - a(\varphi)}$

$$W^*(\varphi) a_k W(\varphi) = a_k + \alpha^{-2}\varphi(k) \quad \text{and} \quad W^*(\varphi) a_k^* W(\varphi) = a_k^* + \alpha^{-2}\overline{\varphi(k)}.$$

Pekar product state: For $\Psi = \psi \otimes W(\alpha^2\varphi)\Omega$ we have

$$a_k \Psi = \varphi(k)\Psi \quad \text{and} \quad a_k^*\Psi = \left(\overline{\varphi(k)} + \mathcal{O}(\alpha^{-1})\right)\Psi.$$

Pekar product states

Weyl operator: $W(\varphi) = e^{a^*(\varphi) - a(\varphi)}$

$W^*(\varphi) a_k W(\varphi) = a_k + \alpha^{-2}\varphi(k)$ and $W^*(\varphi) a_k^* W(\varphi) = a_k^* + \alpha^{-2}\overline{\varphi(k)}$.

Pekar product state: For $\Psi = \psi \otimes W(\alpha^2 \varphi)\Omega$ we have

$$a_k \Psi = \varphi(k)\Psi \quad \text{and} \quad a_k^* \Psi = \left(\overline{\varphi(k)} + \mathcal{O}(\alpha^{-1})\right)\Psi.$$

Ground state energy:

$$\langle \Psi, H\,\Psi \rangle = \int d^3x \; |\nabla \psi(x)|^2 + 2\mathrm{Re} \int d^3k \; |k|^{-1} \langle \psi, e^{ik\cdot x}\psi \rangle \varphi(k) + \|\varphi\|_2^2$$
$$=: \mathcal{E}(\psi, \varphi).$$

Rigorous results:

$$\inf \sigma(H) = \inf_{\psi,\varphi \in L^2(\mathbb{R}^3) : \|\psi\|_2 = 1} \mathcal{E}(\psi,\varphi) + c\alpha^{-2} + o(\alpha^{-2}).$$

[Donsker, Varadhan '83], [Lieb, Thomas '97], [Frank, Seiringer '19].

The Landau-Pekar equations

Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfy

$$\begin{cases} i\partial_t \psi_t(x) & = h_{\varphi_t} \psi_t(x) \\ i\alpha^2 \partial_t \varphi_t(k) & = \varphi_t(k) + \sigma_{\psi_t} \end{cases} \tag{LP}$$

with $(\psi_0, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here, $h_\varphi = -\Delta + V_\varphi$ and

$$V_\varphi = 2\mathrm{Re} \int d^3k \; |k|^{-1} \, e^{ik\cdot x} \varphi(k), \quad \sigma_\psi(k) = (2\pi)^{3/2} \, |k|^{-1} \, \widehat{|\psi|^2}(k).$$

Remarks:

- interaction between an electron and a classical phonon field,

The Landau-Pekar equations

Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfy

$$\begin{cases} i\partial_t \psi_t(x) &= h_{\varphi_t}\psi_t(x) \\ i\alpha^2 \partial_t \varphi_t(k) &= \varphi_t(k) + \sigma_{\psi_t} \end{cases} \tag{LP}$$

with $(\psi_0, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here, $h_\varphi = -\Delta + V_\varphi$ and

$$V_\varphi = 2\mathrm{Re} \int d^3k \, |k|^{-1} \, e^{ik\cdot x} \varphi(k), \quad \sigma_\psi(k) = (2\pi)^{3/2} \, |k|^{-1} \, \widehat{|\psi|^2}(k).$$

Remarks:

- interaction between an electron and a classical phonon field,
- nonlinear equations,

The Landau-Pekar equations

Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfy

$$\begin{cases} i\partial_t \psi_t(x) & = h_{\varphi_t} \psi_t(x) \\ i\alpha^2 \partial_t \varphi_t(k) & = \varphi_t(k) + \sigma_{\psi_t} \end{cases} \tag{LP}$$

with $(\psi_0, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here, $h_\varphi = -\Delta + V_\varphi$ and

$$V_\varphi = 2\mathrm{Re} \int d^3k \; |k|^{-1} \, e^{ik\cdot x} \varphi(k), \quad \sigma_\psi(k) = (2\pi)^{3/2} \, |k|^{-1} \, \widehat{|\psi|^2}(k).$$

Remarks:

- interaction between an electron and a classical phonon field,
- nonlinear equations,
- well-posedness was shown by [Frank, Gang '15],

The Landau-Pekar equations

Let $(\psi_t, \varphi_t) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfy

$$\begin{cases} i\partial_t \psi_t(x) & = h_{\varphi_t}\psi_t(x) \\ i\alpha^2 \partial_t \varphi_t(k) & = \varphi_t(k) + \sigma_{\psi_t} \end{cases} \tag{LP}$$

with $(\psi_0, \varphi_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here, $h_\varphi = -\Delta + V_\varphi$ and

$$V_\varphi = 2\mathrm{Re} \int d^3k \; |k|^{-1} \, e^{ik\cdot x} \varphi(k), \quad \sigma_\psi(k) = (2\pi)^{3/2} \, |k|^{-1} \, \widehat{|\psi|^2}(k).$$

Remarks:

- interaction between an electron and a classical phonon field,
- nonlinear equations,
- well-posedness was shown by [Frank, Gang '15],
- large $\alpha \Rightarrow$ separation of scales.

Main results

Goal: $e^{iHt}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega \approx \psi_t \otimes W(\alpha^2\varphi_t)\Omega$

Main results

Goal: $e^{iHt}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega \approx \psi_t \otimes W(\alpha^2\varphi_t)\Omega$

Main results:

Let $\varphi_0 \in L^2(\mathbb{R}^3, |k|^{1/2}dk)$ such that $e(\varphi_0) = \inf_\psi \langle\psi, h_{\varphi_0}\psi\rangle < 0$ and ψ_{φ_0} denote the unique positive ground state of h_{φ_0} associated to $e(\varphi_0)$.
Let (ψ_t, φ_t) be the solution of (LP) with initial value $(\psi_{\varphi_0}, \varphi_0)$.

[L., Mitrouskas, Rademacher, Schlein, Seiringer (2020)]: Let $\Psi_0 = \psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Omega$, $\gamma_t^{\mathrm{el}} = \mathrm{Tr}_{\mathcal{F}} \left|\mathrm{e}^{-\mathrm{iHt}}\Psi_0\rangle\langle\mathrm{e}^{-\mathrm{iHt}}\Psi_0\right|$ and $\gamma_t^{\mathrm{ph}}(k,l) = \langle e^{-iHt}\Psi_0, a_l^* a_k e^{-iHt}\Psi_0\rangle_{\mathcal{H}}$. Then, there exist $C, T > 0$ such that for all $|t| \leq T\alpha^2$

$$\left\|\gamma_t^{\mathrm{el}} - |\psi_t\rangle\langle\psi_t|\right\|_{\mathrm{tr}} \leq C\alpha^{-1} \text{ and } \left\|\gamma_t^{\mathrm{ph}} - |\varphi_t\rangle\langle\varphi_t|\right\|_{\mathrm{tr}} \leq C\left(\alpha^{-1/4} + \alpha^{-2}\right).$$

Main results

Let $\varphi_0 \in L^2(\mathbb{R}^3)$ such that $e(\varphi_0) < 0$, ψ_{φ_0} be the ground state of h_{φ_0} and (ψ_t, φ_t) be the solution of (LP) with initial value $(\psi_{\varphi_0}, \varphi_0)$.

Bogoliubov dynamics: For $\Upsilon \in \mathcal{F}$ and $t \leq T\alpha^2$, we define

$$i\partial_t \Upsilon_t = (\mathcal{N} - \mathcal{A}_t)\,\Upsilon_t\ , \quad \Upsilon_0 = \Upsilon$$

where $\mathcal{A}_t$ is a quadratic Hamiltonian in Fock-space.

Main results

Let $\varphi_0 \in L^2(\mathbb{R}^3)$ such that $e(\varphi_0) < 0$, ψ_{φ_0} be the ground state of h_{φ_0} and (ψ_t, φ_t) be the solution of (LP) with initial value $(\psi_{\varphi_0}, \varphi_0)$.

Bogoliubov dynamics: For $\Upsilon \in \mathcal{F}$ and $t \leq T\alpha^2$, we define

$$i\partial_t \Upsilon_t = (\mathcal{N} - \mathcal{A}_t)\, \Upsilon_t \;, \quad \Upsilon_0 = \Upsilon$$

where $\mathcal{A}_t$ is a quadratic Hamiltonian in Fock-space.

[L., Mitrouskas, Rademacher, Schlein, Seiringer (2020)]: Let $\langle \Upsilon, \mathcal{N}^5 \Upsilon \rangle_{\mathcal{F}} \leq c\alpha^{-10}$. Then, there exist $C, T > 0$ such that for all $|t| \leq T\alpha^2$

$$\left\| e^{-iHt} \left(\psi_{\varphi_0} \otimes W(\alpha^2 \varphi_0) \Upsilon \right) - e^{-i \int_0^t du\, \omega(u)} \psi_t \otimes W(\alpha^2 \varphi_t) \Upsilon_t \right\| \leq C\alpha^{-1}.$$

Main results

Let $\varphi_0 \in L^2(\mathbb{R}^3)$ such that $e(\varphi_0) < 0$, ψ_{φ_0} be the ground state of h_{φ_0} and (ψ_t, φ_t) be the solution of (LP) with initial value $(\psi_{\varphi_0}, \varphi_0)$.

Bogoliubov dynamics: For $\Upsilon \in \mathcal{F}$ and $t \leq T\alpha^2$, we define

$$i\partial_t \Upsilon_t = (\mathcal{N} - \mathcal{A}_t)\,\Upsilon_t\,, \quad \Upsilon_0 = \Upsilon$$

where $\mathcal{A}_t$ is a quadratic Hamiltonian in Fock-space.

[L., Mitrouskas, Rademacher, Schlein, Seiringer (2020)]: Let $\langle \Upsilon, \mathcal{N}^5 \Upsilon\rangle_{\mathcal{F}} \leq c\alpha^{-10}$. Then, there exist $C, T > 0$ such that for all $|t| \leq T\alpha^2$

$$\left\| e^{-iHt}\left(\psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t du\,\omega(u)}\psi_t \otimes W(\alpha^2\varphi_t)\Upsilon_t \right\| \leq C\alpha^{-1}.$$

Note: $\mathcal{A}_t = \langle \psi_{\varphi_t}, \phi\,(G_\cdot)\, R_{\varphi_t} \phi\,(G_\cdot)\,\psi_{\varphi_t}\rangle_{L^2(\mathbb{R}^3)}$ with $q_{\varphi_t} = 1 - |\psi_{\varphi_t}\rangle\langle\psi_{\varphi_t}|$ and $R_t = q_{\varphi_t}\,(h_{\varphi_t} - e(\varphi_t))\,q_{\varphi_t}$.

Remarks:

For $\Upsilon_0 = \Omega$ and $\delta > 0$ sufficiently small $\exists C_\delta > 0$ s.t for $t = \delta\alpha^2$

$$\left\| e^{-iHt}\left(\psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Omega\right) - e^{-i\int_0^t du\,\omega(u)}\psi_t \otimes W(\alpha^2\varphi_t)\Omega\not{\Upsilon}_{\not{t}} \right\| \geq C_\delta .$$

Remarks:

For $\Upsilon_0 = \Omega$ and $\delta > 0$ sufficiently small $\exists C_\delta > 0$ s.t for $t = \delta\alpha^2$

$$\left\| e^{-iHt} \left(\psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Omega\right) - e^{-i\int_0^t du\, \omega(u)} \psi_t \otimes W(\alpha^2\varphi_t)\Omega \not{\Upsilon}_{\not{t}} \right\| \geq C_\delta .$$

Summary:

- The L.P.-equations approximate well the time evolution of the one-particle reduced density matrices.
- For a norm approximation quantum fluctuations has to be taken into account.
- Non-trivial variations of the phonon field happen over times of order α^2.
- The condition that the initial electron wave function is a ground state of h_{φ_0} is crucial.

Comparison with the literature

$\Psi_0 = \psi_0 \otimes W(\alpha^2\varphi_0)\Omega$, $i\partial\widetilde{\psi}_t = h_{\varphi_0}\widetilde{\psi}_t$:

$$\left.\begin{array}{ll} \text{[Frank, Schlein '13]}: & e^{-iHt}\Psi_0 \approx \widetilde{\psi}_t \otimes W(\alpha^2\varphi_0)\Omega \\ \text{[Frank, Gang '15]}: & e^{-iHt}\Psi_0 \approx \psi_t \otimes W(\alpha^2\varphi_t)\Omega \end{array}\right\} \text{ for } |t| \ll \alpha,$$

$\Psi_{\varphi^P} = \psi^P \otimes W(\alpha^2\varphi^P)\Omega$ with (ψ^P, φ^P) minimizing the Pekar energy;
$\Psi_{\varphi_0} = \psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Omega$ with $(\psi_{\varphi_0}, \varphi_0)$ as in the previous theorem:

$$\left.\begin{array}{ll} \text{[Griesemer '16]}: & e^{-iHt}\Psi_{\varphi^P} \approx e^{-iE_P t}\Psi_{\varphi^P} \\ \text{[L.,R.,S.,S. '19]}: & e^{-iHt}\Psi_{\varphi_0} \approx \psi_t \otimes W(\alpha^2\varphi_t)\Omega \end{array}\right\} \text{ for } |t| \ll \alpha^2,$$

$$\text{[Mitrouskas '20]}: \quad e^{-iHt}\Psi_{\varphi^P} \approx e^{-i\widetilde{H}t}\Psi_{\varphi^P} \quad \text{for } |t| \sim \alpha^2,$$

$$\text{[L.,M.,R.,S.,S. '20]}: \quad e^{-iHt}\Psi_{\varphi_0} \approx \psi_t \otimes W(\alpha^2\varphi_t)\Upsilon_t \quad \text{for } |t| \le T\alpha^2.$$

Sketch of the proof

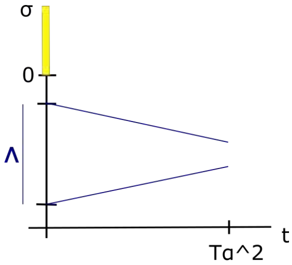
Ingredients of the proof:

- Existence of a spectral gap,
- Adiabatic theorem,
- Duhamel's formula,
- Bogoliubov dynamics.

Sketch of the proof

Ingredients of the proof:

- Existence of a spectral gap,
- Adiabatic theorem,
- Duhamel's formula,
- Bogoliubov dynamics.

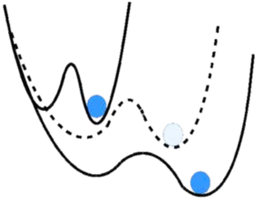


$$h_{\varphi_t} = h_{\varphi_0} - \alpha^{-2} \int_0^t ds\, V_{i\varphi_s}.$$

Sketch of the proof

Ingredients of the proof:

- Existence of a spectral gap,
- Adiabatic theorem,
- Duhamel's formula,
- Bogoliubov dynamics.



L., Rademacher, Schlein, Seiringer (2019): Let (ψ_t, φ_t) be the solution of (LP) with initial value $(\psi_{\varphi_0}, \varphi_0)$. Then there exist $C, T > 0$ s.t.

$$\left\| \psi_t - e^{-i \int_0^t du \, e(\varphi_u)} \psi_{\varphi_t} \right\|_2^2 \leq C\alpha^{-4}(1 + \alpha^{-4}|t|^2) \quad \text{for all} \quad |t| \leq T\alpha^2.$$

Figure: https://medium.com

Sketch of the proof

$$
\begin{aligned}
&\left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)}\psi_t\otimes W(\alpha^2\varphi_t)\Upsilon_t\right\| \\
&\le C\alpha^{-2} + \left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)+e(\varphi_s)}\psi_{\varphi_t}\otimes W(\alpha^2\varphi_t)\Upsilon_t\right\| \\
&\le C\alpha^{-2} + \left\|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\right\|, \quad \text{where}
\end{aligned}
$$

$$
\xi_t = e^{i\int_0^t ds\,\omega(s)+e(\varphi_s)}W^*(\alpha^2\varphi_t)e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right)
$$

Sketch of the proof

$$\left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)}\psi_t\otimes W(\alpha^2\varphi_t)\Upsilon_t\right\|$$
$$\leq C\alpha^{-2} + \left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)+e(\varphi_s)}\psi_{\varphi_t}\otimes W(\alpha^2\varphi_t)\Upsilon_t\right\|$$
$$\leq C\alpha^{-2} + \|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\|\,,\quad \text{where}$$

$$\xi_t = e^{i\int_0^t ds\,\omega(s)+e(\varphi_s)}W^*(\alpha^2\varphi_t)e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right)$$

Duhamel's formula: $\|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\|^2 = -2\mathrm{Re}\int_0^t ds\,\frac{d}{ds}\langle\xi_s,\psi_{\varphi_s}\otimes\Upsilon_s\rangle.$

Sketch of the proof

$$\left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)}\psi_t\otimes W(\alpha^2\varphi_t)\Upsilon_t \right\|$$
$$\leq C\alpha^{-2} + \left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)+e(\varphi_s)}\psi_{\varphi_t}\otimes W(\alpha^2\varphi_t)\Upsilon_t \right\|$$
$$\leq C\alpha^{-2} + \|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\|, \quad \text{where}$$

$$\xi_t = e^{i\int_0^t ds\,\omega(s)+e(\varphi_s)}W^*(\alpha^2\varphi_t)e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right)$$

Duhamel's formula: $\|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\|^2 = -2\mathrm{Re}\int_0^t ds\,\frac{d}{ds}\langle\xi_s, \psi_{\varphi_s}\otimes\Upsilon_s\rangle$.

- $i\partial_s\xi_s = \left(h_{\varphi_s} - e(\varphi_s) + \underbrace{\phi(G_x) - \langle\psi_s, \phi(G_x)\psi_s\rangle}_{\phi(\delta_s G_x)} + \mathcal{N}\right)\xi_s$

Sketch of the proof

$$
\begin{aligned}
&\left\| e^{-iHt}\left(\psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)}\psi_t \otimes W(\alpha^2\varphi_t)\Upsilon_t \right\| \\
&\leq C\alpha^{-2} + \left\| e^{-iHt}\left(\psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)+e(\varphi_s)}\psi_{\varphi_t} \otimes W(\alpha^2\varphi_t)\Upsilon_t \right\| \\
&\leq C\alpha^{-2} + \|\xi_t - \psi_{\varphi_t} \otimes \Upsilon_t\|, \quad \text{where}
\end{aligned}
$$

$$
\xi_t = e^{i\int_0^t ds\,\omega(s)+e(\varphi_s)} W^*(\alpha^2\varphi_t) e^{-iHt}\left(\psi_{\varphi_0} \otimes W(\alpha^2\varphi_0)\Upsilon\right)
$$

Duhamel's formula: $\|\xi_t - \psi_{\varphi_t} \otimes \Upsilon_t\|^2 = -2\mathrm{Re}\int_0^t ds\, \frac{d}{ds}\langle \xi_s, \psi_{\varphi_s} \otimes \Upsilon_s\rangle.$

- $i\partial_s\xi_s = \Big(h_{\varphi_s} - e(\varphi_s) + \underbrace{\phi(G_x) - \langle \psi_s, \phi(G_x)\psi_s\rangle}_{\phi(\delta_s G_x)} + \mathcal{N}\Big)\xi_s$
- $q_{\varphi_s} i\partial_s\xi_s = q_{\varphi_s}\left(h_{\varphi_s} - e(\varphi_s) + \phi\left(\delta_s G_x\right) + \mathcal{N}\right)\xi_s \sim 1 + \alpha^{-1} + \alpha^{-2}$

Sketch of the proof

$$\left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)}\psi_t\otimes W(\alpha^2\varphi_t)\Upsilon_t\right\|$$
$$\leq C\alpha^{-2} + \left\| e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right) - e^{-i\int_0^t ds\,\omega(s)+e(\varphi_s)}\psi_{\varphi_t}\otimes W(\alpha^2\varphi_t)\Upsilon_t\right\|$$
$$\leq C\alpha^{-2} + \|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\|\,, \quad \text{where}$$

$$\xi_t = e^{i\int_0^t ds\,\omega(s)+e(\varphi_s)}W^*(\alpha^2\varphi_t)e^{-iHt}\left(\psi_{\varphi_0}\otimes W(\alpha^2\varphi_0)\Upsilon\right)$$

Duhamel's formula: $\|\xi_t - \psi_{\varphi_t}\otimes\Upsilon_t\|^2 = -2\mathrm{Re}\int_0^t ds\,\frac{d}{ds}\langle\xi_s, \psi_{\varphi_s}\otimes\Upsilon_s\rangle$.

- $i\partial_s\xi_s = \Big(h_{\varphi_s} - e(\varphi_s) + \underbrace{\phi(G_x) - \langle\psi_s, \phi(G_x)\psi_s\rangle}_{\phi(\delta_s G_x)} + \mathcal{N}\Big)\xi_s$
- $q_{\varphi_s} i\partial_s\xi_s = q_{\varphi_s}\big(h_{\varphi_s} - e(\varphi_s) + \phi(\delta_s G_x) + \mathcal{N}\big)\xi_s \sim 1 + \alpha^{-1} + \alpha^{-2}$
- The time derivatives of all other quantities are of order α^{-2}.
- $q_{\varphi_s}\xi_s = R_s\big(i\partial_s - \phi(\delta_s G_x) - \mathcal{N}\big)\xi_s$ with $R_s = q_{\varphi_s}(h_{\varphi_s} - e(\varphi_s))^{-1}q_{\varphi_s}$.

Related results

Adiabatic theorem:

[Frank '17], [Frank, Gang '19].

Classical limit of quantum fields:

[Davies '79], [Ginibre, Velo '79] [Hiroshima '98], [Teufel '02], [Ginibre, Nironi, Velo '06], [Falconi '12], [Ammari, Falconi '16], [L., Pickl '16], [Correggi, Falconi '17], [Corregi, Falconi, Olivieri '18], [L., Pickl '18], [L., Petrat '18], [Carlone, Correggi, Falconi, Olivieri '19], [L., Mitrouskas, Seiringer '20].

Effective mass ($m \sim \alpha^4$):

[Landau, Pekar '48], [Feynman '54], [Lieb, Seiringer '14], [Lieb, Seiringer '19], [Dybalski, Spohn '19].

Thank you for your attention!

Simple example

Let $f \in C^1(\mathbb{R}, \mathbb{R})$, $c, d, e \in \mathbb{R}$ with $c > 0$ and $\omega = c + \alpha^{-1} d + \alpha^{-2} e$.

$$\begin{aligned}
&\int_0^t ds\, f(\alpha^{-2} s) e^{-i\omega s} \\
&= c^{-1} \int_0^t ds\, f(\alpha^{-2} s) \Big(i \frac{d}{ds} - \alpha^{-1} d - \alpha^{-2} e \Big) e^{-i\omega s} \\
&= \text{b.t.} + \mathcal{O}(\alpha^{-2} t) - \alpha^{-1} c^{-1} d \int_0^t ds\, f(\alpha^{-2} s) e^{-i\omega s} \\
&= \text{b.t.} + \mathcal{O}(\alpha^{-2} t) - \alpha^{-1} c^{-2} d \int_0^t ds\, f(\alpha^{-2} s) \Big(i \frac{d}{ds} - \alpha^{-1} d - \alpha^{-2} e \Big) e^{-i\omega s} \\
&= \text{b.t.} + \mathcal{O}(\alpha^{-2} t).
\end{aligned}$$