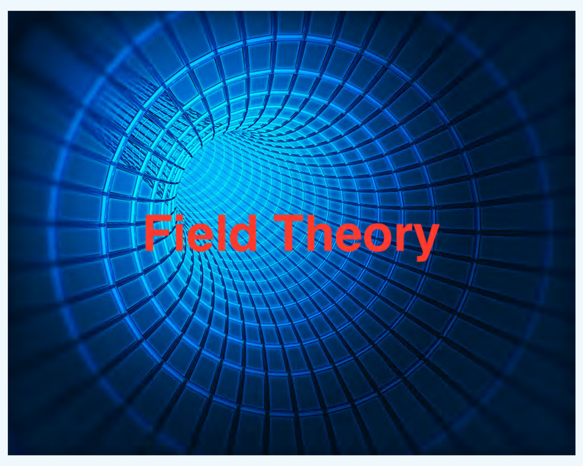

Quantization of Dubrovin Connections

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Motivations

The Dubrovin systems (or Frobenius manifolds) give a geometrical formulation of Witten-Dijkgraaf-Verlinde-Verlinde equations governing deformations of 2D topological field theories. It plays key roles in the study of Gromov-Witten theory and integrable hierarchies.

The monodromy data of a Dubrovin system, which classifies the system, is a Stokes matrix. Ugaglia identifies the space of the Stokes matrices with a Poisson homogenous space [6], the semiclassical limit of a quantum symmetric pair [5]. Thus we hope to see the role of the quantum symmetric pair in the theory of Dubrovin systems.

Contribution

In the poster, we propose a quantization of the Dubrovin systems of Frobenius manifolds, via an isomonodromy deformation of the cyclotomic Knizhnik–Zamolodchikov (IKZ) systems, and then explore its relation with quantum symmetric pairs and Givental twisted loop groups [4]. We expect that the quantization has more applications in Gromov-Witten type theory and the related theory of integrable hierachies.

References

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Dubrovin systems

We consider a linear system for a matrix valued function $F(z,u^1,...,u^n)$

$$\frac{\partial F}{\partial z}=\left(\frac{u}{z^2}+\frac{V(u)}{z}\right)F,$$
$$\frac{\partial F}{\partial u^i}=V_i(z,u)\cdot F.$$

Here $u=\mathrm{diag}(u^1,...,u^n)$, the matrix function $V_i(z,u)$ is determined by $V(u)$, and $V(u)$ satisfies a differential equation (compatibility of the system).

Stokes matrix and isomonodromy:
Fix u, the first equation has a canonical fundamental solution in each Stokes sector on z-plane. Take two opposite sectors $\mathrm{D}_\pm$ and the associated solutions $F_\pm$.

Stokes matrices $S_\pm(u)$ are determined by

$$F_-=F_+\cdot S_+(u),\quad F_+=F_-\cdot S_-(u)$$

where the first (second) identity is hold in D_- (D_+).

Isomonodromy: $S_\pm(u)$ don't rely on u.

Thus the defining equation of $V(u)$ (compatibility for the Dubrovin system) describes an isomonodromy deformation.

Isomonodromy KZ systems

We introduce a linear system for a $U(\mathrm{so}_n)\widehat{\otimes}U(\mathrm{gl}_n)[\![\hbar]\!]$–valued function $F(z,u^1,...,u^n)$:

$$\frac{\partial F_\hbar}{\partial z}=\left(\frac{u\otimes 1}{z^2}+\hbar\frac{\Omega(u)}{z}\right)F_\hbar,$$
$$\frac{\partial F_\hbar}{\partial u^i}=\Omega_i(z,u,\hbar)\cdot F_\hbar.$$

Here the function $\Omega_i(z,u)$ is determined by $\Omega(u)$, and $\Omega(u)$ satisfies a differential equation (compatibility of the system).

Quantum Stokes matrix and isomonodromy:
Fix u, the first equation has a canonical solution in each Stokes sector on z-plane. Take two opposite sectors $\mathrm{D}_\pm$ and the associated solutions $F_{\hbar\pm}$.

q-Stokes matrices $S_{\hbar\pm}(u)\in U(\mathrm{so}_n)\widehat{\otimes}U(\mathrm{gl}_n)[\![\hbar]\!]$:

$$F_{\hbar-}=F_{\hbar+}\cdot S_{\hbar+}(u),\quad F_{\hbar+}=F_{\hbar-}\cdot S_{\hbar-}(u)$$

where the first (second) identity is hold in D_- (D_+).

Theorem 1 (Isomonodromy). *$S_{\hbar\pm}(u)$ don't rely on u.*

Thus the defining equation of $\Omega(u)$ (compatibility for the IKZ system) describes a quantum isomonodromy deformation.

Quantization of Dubrovin systems

$$\begin{array}{ccccc}
\text{Symplectic actions on } \mathcal{H}[\![\hbar]\!] & \xleftarrow{\text{Solutions}} & \text{IKZ systems} & \xrightarrow{q-\text{Stokes matrices}} & \text{Quantum symmetric pairs} \\
\downarrow \hbar=0 & & \downarrow \hbar=0 & & \downarrow \hbar=0 \\
\text{Symplectic actions on } \mathcal{H} & \xleftarrow{\text{Solutions}} & \text{Dubrovin systems} & \xrightarrow{\text{Stokes matrices}} & \text{Poisson homogenous spaces}
\end{array}$$

Semiclassical limit: a way of letting $\hbar=0$ in the literature of quantum algebras [2].

Theorem 2. *The semiclassical limit of the IKZ system gives rise to the Dubrovin systems. In particular, any solution F of a Dubrovin system has a natural deformation $F_\hbar=F+\hbar F_1+O(\hbar^2)$.*

Reflection equations and Gromov-Witten type theory

(a). q-Stokes matrices and reflection equations [1].

Theorem 3. *The q-Stokes matrices $S_{\hbar\pm}$ of the IKZ system satisfy the reflection equation.*

$$S^{(12)}_{\hbar\pm}R^{32}S^{(13)}_{\hbar\pm}R^{32}=R^{32}S^{(13)}_{\hbar\pm}R^{23}S^{(12)}_{\hbar\pm}.$$

Here R is certain quantum R-matrix.

Furthermore, the monodromy data of the IKZ system can be used to construct the quantum symmetric pair introduced by [5], where the q-Stokes matrices play the role of the universal K-matrix.

(b). Stokes matrices and Poisson homogenous spaces.
On the other hand, the semiclassical limit of the q-Stokes matrices $S_{\hbar\pm}$ of the IKZ system are the Stokes matrices $S_\pm$ of the Dubrovin systems. As a corollary, we obtain a commutative diagram as in the above picture. In particular, it gives a "quantum" interpretation of

Theorem 4. [6] *The space of Stokes matrices is a Poisson homogenous space.*

(c). Deformation of Givental twisted loop group.
For fixed u, solutions $F(z)$ of the Dubrovin system can be viewed as symplectic transformations on certain symplectic vector space H. The natural $\hbar$-deformation $F_\hbar=F+F_1\hbar+F_2\hbar^2+\cdots$ of F, via the IKZ system, is viewed as a linear transformation on $\mathcal{H}[\![\hbar]\!]$.

Theorem 5. *$F_\hbar=F+O(\hbar)$ is symplectic, i.e., an $\hbar$-deformation of the symplectic transformation F on $\mathcal{H}$.*

Solutions of the Dubrovin systems have two deformation/quantization:
- $\hbar$-deformation via the IKZ system;
- ε-deformation via Givental quantization [4].

Theorem 5 enables us to combine these two into one quantization with two parameters. In terms of integrable hierarchies, the two parameters ε and $\hbar$ may correspond respectively to the dispersion and quantization parameters.

All the theorems in this poster will be included in our paper [8] and a new version of [7].