$\mathcal{N} = 4$ Gauge Theory and Planar Integrability

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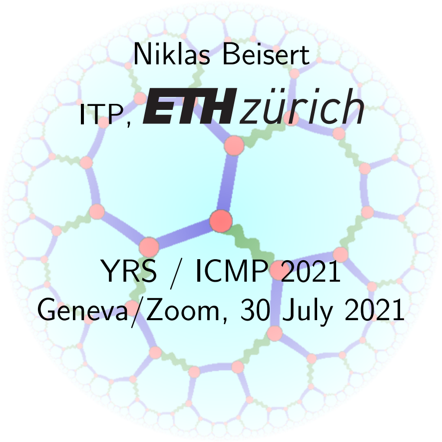

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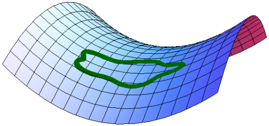
I. Introduction

AdS/CFT Correspondence

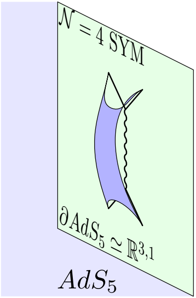
Want to understand strings on curved target spaces:

- non-linear equations
- spectrum difficult
- scattering?! how to get started?



Major achievement: conjectured exact **AdS/CFT duality** [Maldacena hep-th/9711200]

- string (gravitational) theory on AdS target space,
- conformal field theory (CFT) on boundary of AdS.



Prototype duality:

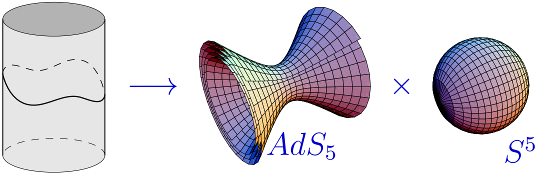
- IIB strings on $AdS_5 \times S^5$ target space
- $\mathcal{N} = 4$ supersymmetric Yang–Mills (4D CFT)

Features:

- highly symmetric, highly accessible;
- but: non-linear models, strong/weak duality.

Strings on $AdS_5 \times S^5$

IIB superstrings on curved $AdS_5 \times S^5$ space:



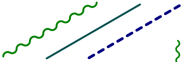
- 2D non-linear sigma model (QFT),
- worldsheet coupling λ,
- string coupling: g_s,

$$\mathcal{S} \sim \sqrt{\lambda} \int (\mathrm{d}\xi)^2 \big((\mathrm{D}X)^2 + \text{fermions}\big)$$

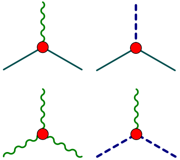
- weakly coupled for large λ,
- symmetry: background isometries $\widetilde{\mathrm{PSU}}(2,2|4)$.

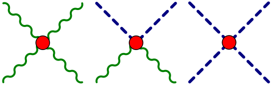
$\mathcal{N}=4$ Super Yang–Mills Theory

4D Quantum Field Theory Model: [Brink Schwarz Scherk]



- gauge field A_μ, 4 fermions Ψ, 6 scalars Φ.
- gauge group typically $\mathrm{SU}(N_\mathrm{c})$
- all fields massless and adjoint ($N_\mathrm{c} \times N_\mathrm{c}$ matrix)
- standard couplings: non-abelian gauge, Yukawa, Φ^4
- coupling constant g_YM, topological angle θ
- exact superconformal symmetry $\widetilde{\mathrm{PSU}}(2,2|4)$





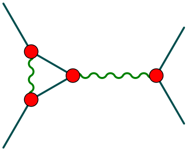
$$\mathcal{S} \sim \frac{1}{g_\mathrm{YM}^2} \int (\mathrm{d}x)^4 \,\mathrm{Tr}\Big(F^2 + (D\Phi)^2 + \bar{\Psi} D \Psi + [\Phi,\Phi]^2 + \mathrm{Re}\,\Psi[\Phi,\Psi]\Big)$$

Supersymmetry helps:

- protects some quantities, e.g. $\beta = 0$,
- but still model far from trivial!

Weakly coupled for small g_YM

compute by Feynman graphs (hard!)

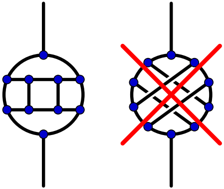


Planar Limit

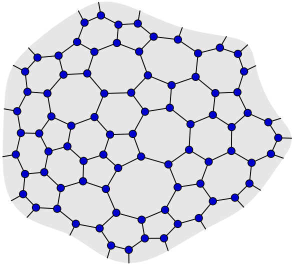
Planar limit in gauge theory:

['t Hooft Nucl. Phys. B72, 461]

- large-N_c limit: $N_\mathrm{c} = \infty$, $g_\mathrm{YM} = 0$,
  't Hooft coupling $g^2_\mathrm{YM} N_\mathrm{c} =: \lambda$ remains,
- only planar Feynman graphs,
  no crossing propagators,
- drastic combinatorial simplification.

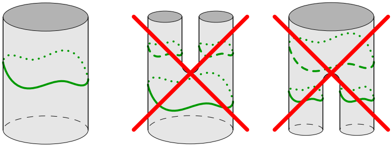


Surface of Feynman graphs becomes 2D string worldsheet:



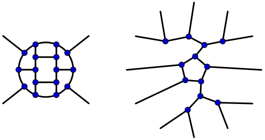
Planar limit in string theory:

- no string coupling $g_\mathrm{s} = 0$,
  no string splitting or joining.
- worldsheet coupling λ remains.



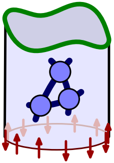
Integrability

Standard QFT approach: **Feynman graphs**



- enormously difficult at higher loops ...
- ... but also at lower loops and many legs.

Planar $\mathcal{N} = 4$ SYM is **integrable** ... see review collection [NB et al. 1012.3982]



- ... so is the AdS/CFT dual string theory.
- integrability vastly simplifies calculations.
- spectrum of local operators now largely understood.
- can compute observables at finite coupling λ. [NB, Eden Staudacher]
- simple integral equation for cusp dimension $D_{\text{cusp}}(\lambda)$.

Local, gauge-invariant operators, e.g. / dual to string states:

$$\mathcal{O} = \operatorname{Tr} \mathcal{D}^{n_1}\Phi\, \mathcal{D}^{n_2}\Phi \cdots \mathcal{D}^{n_L}\Phi \quad \longleftrightarrow$$

Observable: scaling dimension $D_{\mathcal{O}}$ / dual to energy of string state

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle \sim |x-y|^{-2D_{\mathcal{O}}}.$$

II. Planar AdS/CFT Spectrum using Integrability

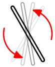
Cusp Dimension from Bethe Equations

Asymptotic Bethe equations for scaling dimensions [Minahan Zarembo][NB Staudacher][Serban Staudacher]

$$\exp(ip_k L) = \prod_{j=1, j\neq k}^{M} S(p_k, p_j; \lambda), \qquad D_{\mathcal{O}} = D_0 + \sum_{j=1}^{M} D(p_k; \lambda).$$

Coupling constant λ enters analytically. [NB, Dippel Staudacher][NB Staudacher]

Useful object: Twist-two operators with spin S / dual spinning string:

 : $\mathcal{O}_S \simeq \operatorname{Tr} \Phi \overleftrightarrow{\mathcal{D}}^S \Phi \qquad \longleftrightarrow$ 

A lot is known about their anomalous dimensions δD_S:

- QCD: δD_S responsible for scale violations in DIS.
- DGLAP, BFKL evolution equations.
- Large-S behaviour: cusp dimension D_{cusp}

$$D_S = D_{\text{cusp}} \log S + \ldots .$$

Cusp Dimension

Cusp dimension determined by AdS/CFT planar integrable system!
Compute cusp dimension using Bethe equations. Integral eq.: [Eden Staudacher]

$$\psi(x) = K(x,0) - \int_0^\infty K(x,y)\,\frac{\mathrm{d}y\,y}{e^{2\pi y/\sqrt{\lambda}}-1}\,\psi(y).$$

Kernel $K = K_0 + K_1 + K_\mathrm{d}$ made from Bessel $\mathrm{J}_{0,1}$ with [NB, Eden Staudacher]

$$K_0(x,y) = \frac{x\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - y\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_1(x,y) = \frac{y\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - x\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_\mathrm{d}(x,y) = 2\int_0^\infty K_1(x,z)\,\frac{\mathrm{d}z\,z}{e^{2\pi z/\sqrt{\lambda}}-1}\,K_0(z,y).$$

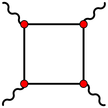
Cusp anomalous dimension: $D_\mathrm{cusp} = (\lambda/\pi^2)\psi(0)$.

Weak/Strong Expansion

Weak-coupling solution of integral equation [NB, Eden Staudacher]

$$D_{\text{cusp}}(\lambda) = \frac{1}{2}\,\frac{\lambda}{\pi^2} - \frac{1}{96}\,\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\,\frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2} \pm \ldots.$$

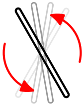
Confirmed by gluon scattering amplitudes [Bern Dixon Smirnov][Bern, Czakon, Dixon Kosower, Smirnov]



$$A(p,\lambda) \simeq A^{(0)}(p)\exp\bigl(2D_{\text{cusp}}(\lambda)M^{(1)}(p)\bigr).$$

Connection between integrability & scattering amplitudes? later...

Strong-coupling asymptotic solution of integral equation [Casteill Kristjansen][Basso Korchemsky Kotański]



$$E_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{\beta(2)}{\pi\sqrt{\lambda}} + \ldots.$$

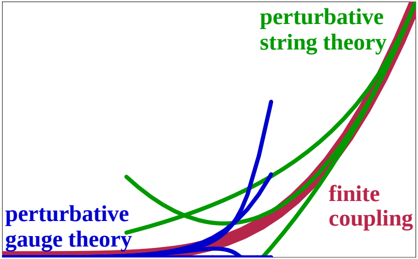
Agreement with energy of spinning string. [Gubser Klebanov Polyakov][Frolov Tseytlin][Roiban Tirziu Tseytlin]

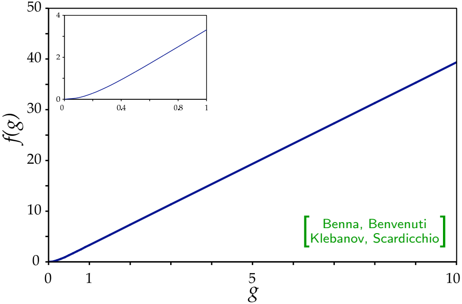
Finite-Coupling Interpolation

Cusp dimension can be computed numerically at finite coupling λ.

Smooth interpolation between perturbative gauge and string theory

- in the Bethe equations (left), [NB, Eden Staudacher]
- in the cusp dimension (right). [Benna, Benvenuti Klebanov, Scardicchio]



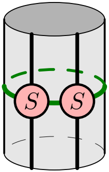


An exact result in a (planar) 4D gauge theory at finite coupling.

Thermodynamic Bethe Ansatz

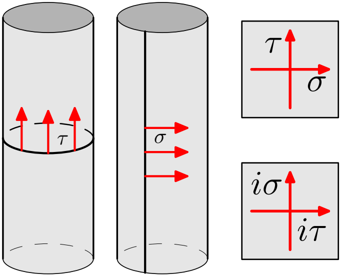
Bethe equations not exact for finite size?

- scattering assumes infinite worldsheet,
- actual string states defined on finite cylinder,
- Lüscher terms: virtual particles around cylinder.



Thermodynamic Bethe Ansatz:

- idea: space has finite extent,
  but time is infinite.
- consider evolution in space,
  scattering problem on infinite line.
- in 2D: double Wick rotation.
  Same S-matrix.

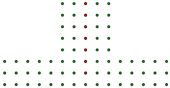


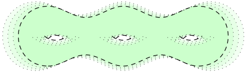
Obtain infinite set of coupled integral equations.

Techniques and Applications

Arsenal of improved integrable techniques:

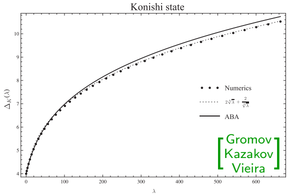
- T/Y-System
- Hirota equations
- Baxter equations
- quantum curves
- finite non-linear integral equations





Consider now particular state ("Konishi"), e.g.

$$\mathcal{O} = \operatorname{Tr} \Phi_m \Phi_m .$$



[Gromov Kazakov Vieira]

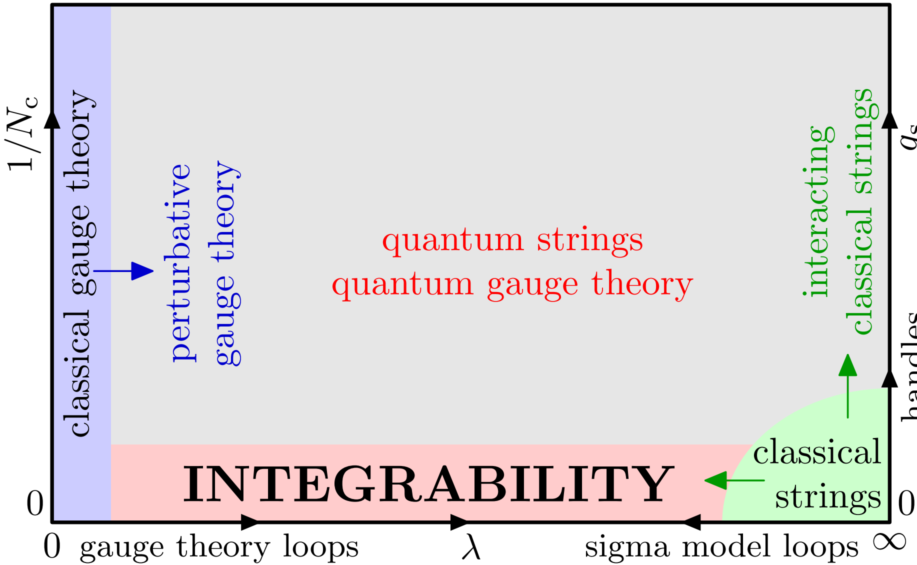
Can now compute the dimension or energy:

- interpolation from weak to strong coupling,
- 8 loops: sum of (multiple) zeta values

[Bajnok Janik] [Leurent Volin]

$$D = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} - \frac{\big(78 - 18\zeta(3) + 45\zeta(5)\big)\lambda^4}{2048\pi^8} + \ldots .$$

Charted Territory

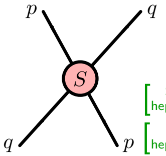


III. Selected Achievements using Integrability

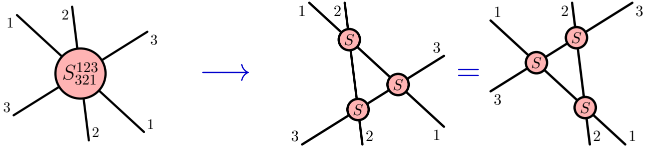
Worldsheet Scattering

Integrability methods rely on scattering picture for 2D worldsheet.

- 8 bosonic + 8 fermionic excitations,
- $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ residual symmetry,
- 2-particle scattering matrix S,
- $S(p, q; \lambda)$ at finite λ determined by symmetry,
- integrability: factorised multi-particle scattering, YBE.



[Staudacher hep-th/0412188]
[NB hep-th/0511082]



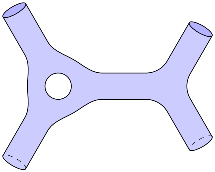
Unusual non-local **symmetry**:

- infinite-dimensional quantum algebra: Yangian
- novel deformation of Yangian $\mathrm{Y}[\mathfrak{u}(2|2)]$ (maths)
- investigations of algebra ongoing

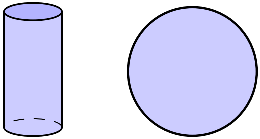
[Gomez Hernández] [Plefka Spill Torrielli]
[NB 0704.0400]
[Matsumoto Moriyama Torrielli] [NB Spill]
[. . .]

Correlation Functions

Integrable methods to compute correlation functions efficiently?

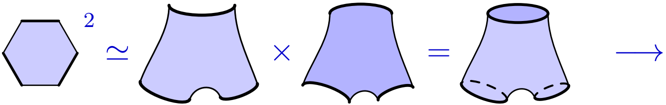


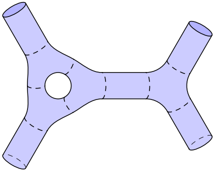
But: Integrability applies to annulus and disc topology only!



Stitch together two hexagons to a pair of pants, then glue arbitrary correlator (as in string theory):

[Escobedo, Gromov Sever, Vieira] [Basso Komatsu Vieira] [. . .]

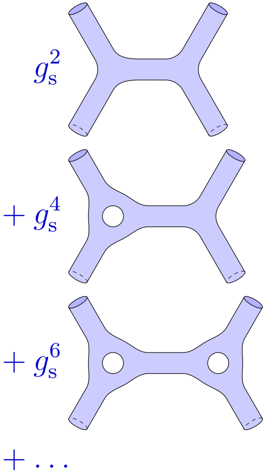




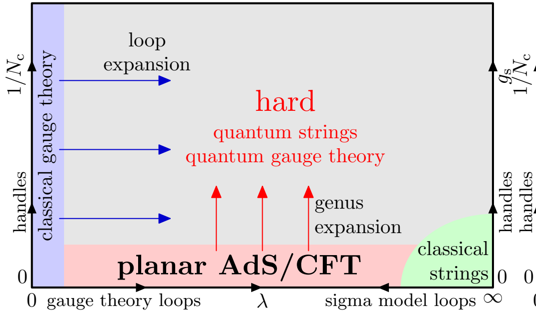
Excitations propagate on/around 2D worldsheet: scattering!

Genus Expansion

Outlook: Genus expansion can now be done by gluing (in principle)



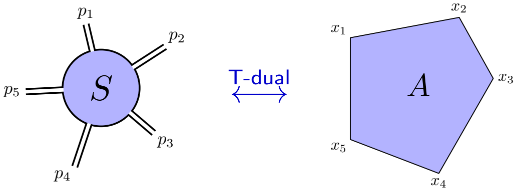
Alternative expansion scheme for $\mathcal{N} = 4$ SYM theory:



Dual Conformal and Yangian Symmetry

Strings on $AdS_5 \times S^5$ self-dual under T-duality. [Alday Maldacena] [Berkovits Maldacena] [NB, Ricci Tseytlin, Wolf]

AdS/CFT: T-Duality for planar $\mathcal{N} = 4$ SYM: [Drummond Korchemsky Sokatchev] [Brandhuber Heslop Travaglini]



- planar colour-ordered scattering amplitudes
- null polygonal Wilson loops

Ordinary and dual conformal combine to **Yangian symmetry**. [Drummond Henn Plefka]

Yangian invariance of the action: $\widehat{\mathrm{J}}\,\mathcal{S} = 0$. [NB, Garus, Rosso 1803.06310]

IV. Summary and Outlook

Status of AdS/CFT Integrability

implications understood for several observables
applied to compute at finite coupling
well-defined math concepts at leading weak coupling
some perturbative corrections under control
understood well in classical string theory on $AdS_5 \times S^5$
new kind of symmetry for planar $\mathcal{N} = 4$ SYM

Open Questions:

What is integrability?
How to define it at higher loops and finite coupling?