Advances in integrable η-deformations of superstrings

Fiona Seibold

ETH zürich

SwissMAP
The Mathematics of Physics
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1. **Integrability**
   - Hamiltonian mechanics, 2D field theories, string theory
2. ***q*-deformations**
   - Drinfel'd Jimbo quantum group, case of superalgebras
3. ***q*-deformations of $AdS_5 \times S^5$**
   - string theory? quantum integrability?
4. **Conclusions and outlook**

1 Integrability

A theory is integrable when there are “enough conserved charges”

A theory is integrable when there are "enough conserved charges"

Hamiltonian system with N d.o.f and N independent conserved quantities F_j in involution

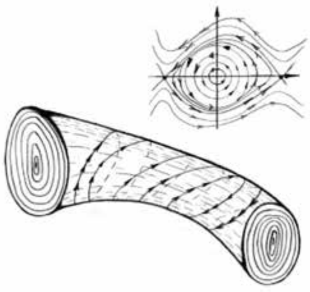
$$\{F_j, F_k\} = 0, \qquad j,k = 1,\dots N$$

→ Liouville integrable

→ Can be solved exactly

→ Lax pair (L, M)

$$\text{e.o.m} \quad \Leftrightarrow \quad \frac{\mathrm{d}L}{\mathrm{d}t} - [M, L] = 0$$



The Liouville Arnol'd theorem, Alexei Tsygvintsev

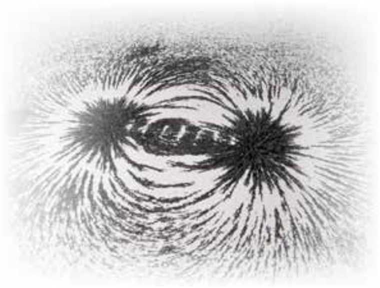
A theory is integrable when there are “enough conserved charges”

Field theories have ∞ many d.o.f

For 2D Field theories:

$\rightarrow$ Lax pair

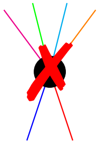
$\rightarrow$ Factorised scattering



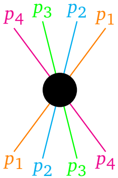
$$\text{In 1D:} \qquad \text{e.o.m} \quad \Leftrightarrow \quad \frac{\mathrm{d}L}{\mathrm{d}t} - [L, M] = 0$$

$$\text{In 2D:} \qquad \text{e.o.m} \quad \Leftrightarrow \quad \partial_\tau \mathscr{L}_\sigma - \partial_\sigma \mathscr{L}_\tau - [\mathscr{L}_\sigma, \mathscr{L}_\tau] = 0$$

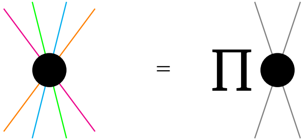
- No particle production

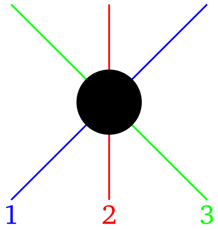


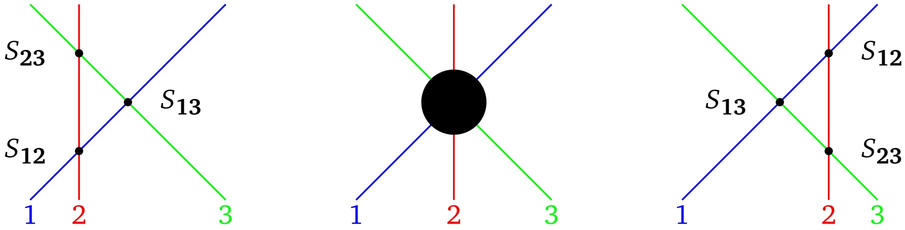
- Transmitted momenta

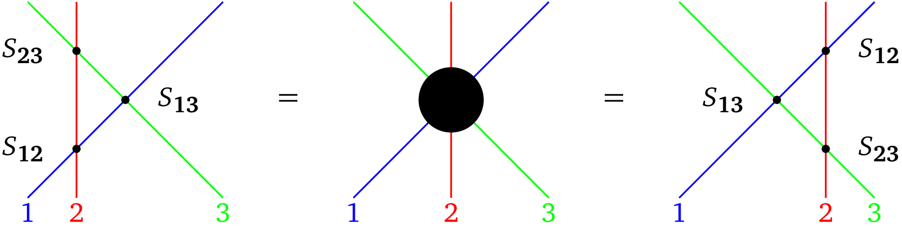


- Factorisation




1
2
3

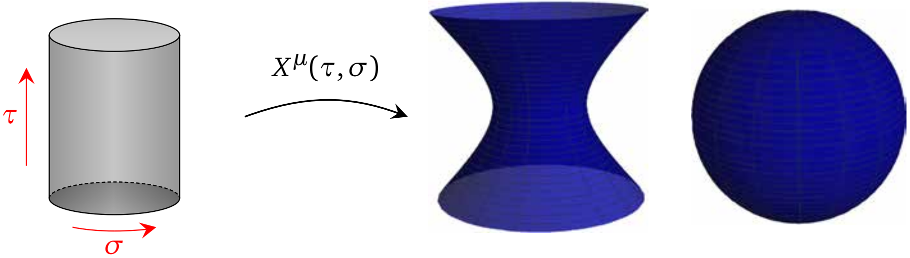

S_{23}
S_{13}
S_{12}
1
2
3
1
2
3
S_{12}
S_{13}
S_{23}
1
2
3



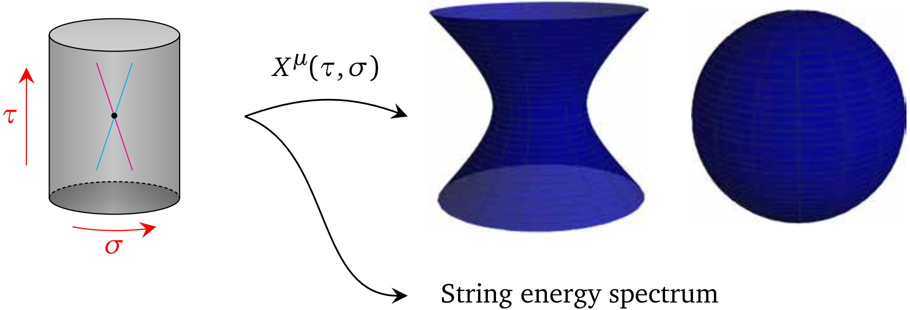
$$S_{23}S_{13}S_{12} \quad = \quad S_{12}S_{13}S_{23}$$

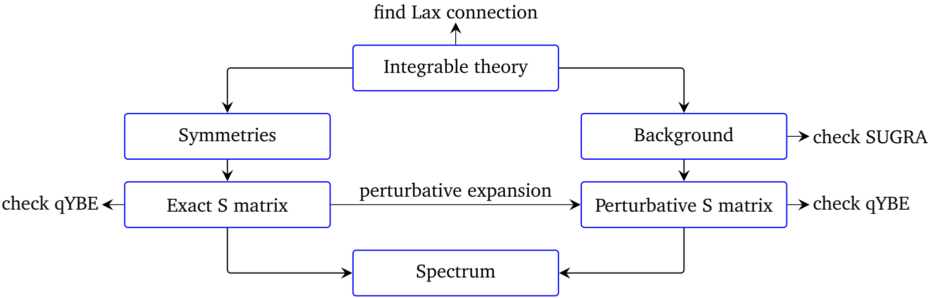
quantum Yang-Baxter equation

String theory as a 2D sigma model



String theory as a 2D sigma model





- Applied with great success for $AdS_5 \times S^5$ superstrings, ...
- Goal: go beyond this most supersymmetric case!

2 q-deformations

- Lie algebras are rigid objects that do not admit deformations
- Idea: consider larger structures

  Universal enveloping algebras
  Hopf algebras

$\Rightarrow$ Drinfel'd Jimbo type quantum group

$\mathscr{A}$	vector space over a field K
μ	product
η	unit
Δ	coproduct
ϵ	counit
S	antipode map

$\mathscr{A}$ is a Hopf algebra if

- $(\mathscr{A}, \mu, \eta)$ is an associative algebra
- $(\mathscr{A}, \Delta, \epsilon)$ is a coalgebra
- $\mu(S \otimes \mathrm{id})\Delta(X) = \mu(id \otimes S)\Delta(X) = \eta\epsilon(X)$

- Any Lie algebra can be promoted to a Hopf algebra

$$[H_j, E_k] = A_{jk}E_k \qquad \Delta(H_j) = H_j \otimes 1 + 1 \otimes H_j \qquad S(H_j) = -H_j$$

$$[H_j, F_k] = -A_{jk}F_k \qquad \Delta(E_j) = E_j \otimes 1 + 1 \otimes E_j \qquad S(E_j) = -E_j$$

$$[E_j, F_k] = \delta_{jk}H_k \qquad \Delta(F_j) = F_j \otimes 1 + 1 \otimes F_j \qquad S(F_j) = -F_j$$

- The associated Drinfel'd Jimbo quantum group is

$$[H_j, E_k] = A_{jk}E_k \qquad \Delta(H_j) = H_j \otimes 1 + 1 \otimes H_j \qquad S(H_j) = -H_j$$

$$[H_j, F_k] = -A_{jk}F_k \qquad \Delta(E_j) = E_j \otimes 1 + q^{-H_j} \otimes E_j \qquad S(E_j) = -q^{H_j}E_j$$

$$[E_j, F_k] = \delta_{jk}\frac{q^{H_j} - q^{-H_j}}{q - q^{-1}} \qquad \Delta(F_j) = F_j \otimes q^{H_j} + 1 \otimes F_j \qquad S(F_j) = -F_j q^{-H_j}$$

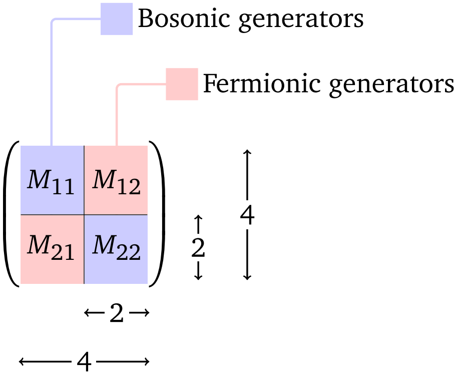
q-deformations

[Klimcik '02 '08]
[Delduc Magro Vicedo '13 '14]
[Sfetsos '13]
[Hollowood, Miramontes and Schmidtt '14]
...

$q \in \mathbb{R}$ → "η" deformations

$q \in e^{i\mathbb{R}}$ → "λ" deformations

- Example: $\mathfrak{g} = \mathfrak{sl}(2|2)$



$$\mathrm{STr}[M] = \mathrm{Tr}[M_{11}] - \mathrm{Tr}[M_{22}] = 0$$

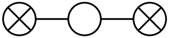
Simple roots

$$\left(\begin{array}{cc|cc} 0 & \textcircled{+} & + & + \\ - & 0 & \textcircled{+} & + \\ \hline - & - & 0 & \textcircled{+} \\ - & - & - & 0 \end{array}\right)$$

Cartan matrix

$$\begin{pmatrix} -2 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +2 \end{pmatrix}$$

Dynkin diagram

Simple roots	$\left(\begin{array}{cc\|cc} 0 & \oplus & + & + \\ - & 0 & \oplus & + \\ \hline - & - & 0 & \oplus \\ - & - & - & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} 0 & \oplus & - & + \\ - & 0 & - & \oplus \\ \hline \oplus & + & 0 & + \\ - & - & - & 0 \end{array}\right)$	$\left(\begin{array}{cc\|cc} 0 & + & \oplus & + \\ - & 0 & - & \oplus \\ \hline - & \oplus & 0 & + \\ - & - & - & 0 \end{array}\right)$
Cartan matrix	$\begin{pmatrix} -2 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & +2 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & 0 \\ +1 & -2 & +1 \\ 0 & +1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
Dynkin diagram			

- Effect of the choice of inequivalent CW bases for superalgebras

$$(\mathfrak{g}, \text{CW}) \qquad (\mathfrak{g}, \text{CW'})$$

- The associative algebras are isomorphic

$$[\omega(X), \omega(Y)] = \omega([X, Y])$$

- The coproducts are related by a twist

$$(\omega \otimes \omega)\Delta(X) = F^{-1}\Delta'(\omega(X))F$$

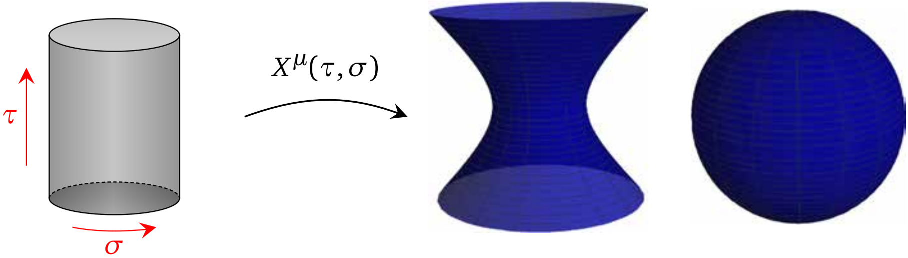
- What are the physical implications of this twist?

  → string theory?
  → spectrum?

3 q-deformations* of the $AdS_5 \times S^5$ superstring

* $q \in \mathbb{R}$

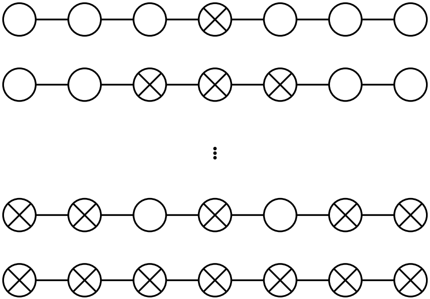
String theory as a 2D sigma model



$$AdS_5 = \frac{SO(2,4)}{SO(1,4)} = \frac{SU(2,2)}{SO(1,4)} \qquad S^5 = \frac{SO(6)}{SO(5)} = \frac{SU(4)}{SO(5)}$$

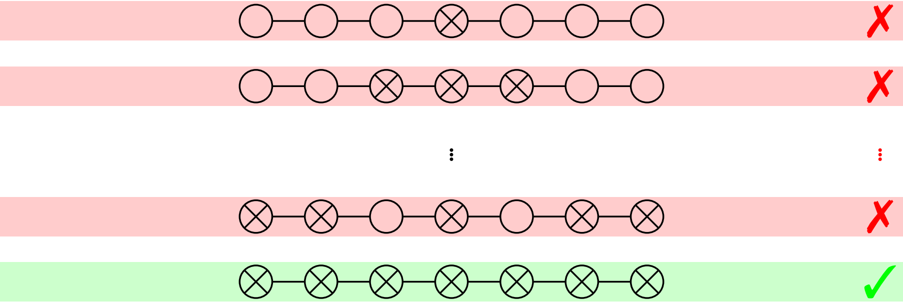
$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$$

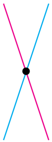
- Symmetry algebra $\mathfrak{psu}(2,2|4)$ has many Dynkin diagrams



- Symmetry algebra $\mathfrak{psu}(2,2|4)$ has many Dynkin diagrams
- String theory?

[Hoare Seibold '18]

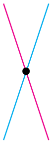




$$\Delta^{op}(X)S = S\Delta(X)$$

$$S' = F^{op}SF^{-1}$$

Effect of twist on the S matrix



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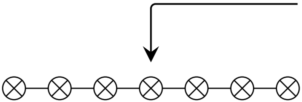
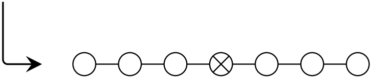
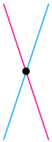




[Beisert, Koroteev '08]
[Arutyunov, Borsato, Frolov '15]
[Seibold, van Tongeren, Zimmermann '20]



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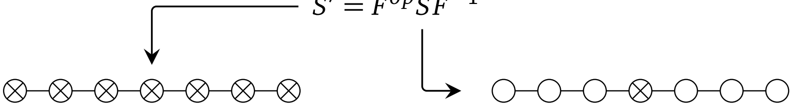

- qYBE satisfied in both cases
- expansion matches the perturbative calculation in both cases

4 Conclusions & outlook

- An efficient way to generate new integrable theories is to deform already known ones
- A particular type of such deformations promotes the symmetry algebra to a quantum group: q-deformations
- NLSM realisation: η-deformations ($q \in \mathbb{R}$), λ-deformations ($q \in e^{i\mathbb{R}}$)
- η-deformations are not unique for superalgebras
- Only one type of η-deformation is a string theory
- The exact S matrices are related by a twist

- How does the twist affect physical observables: spectrum, ...
- Better understand the Weyl anomaly in η-deformations
- Connections between η and λ deformations: Poisson-Lie duality
- Understand q-deformations in the context of holography

Thank You!

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