Existence and Uniqueness of Exact WKB Solutions of Schrödinger Equations

arXiv:2008.06492

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The One-Dimensional Stationary Schrödinger Equation

$$\hbar^2 \psi''(x) - q(x)\psi(x) = 0$$

- potential $q(x) = V(x) - E$
- rarely can be solved in closed form (harmonic oscillator, hydrogen atom, ...)
- $\Rightarrow$ need approximation methods
- want solutions that 'know' about the classical limit

The WKB Approximation Method

Origins: Liouville, Green (1837); Jeffreys (1923);
Wentzel, Kramers, Brillouin (1926)...

Basic Idea: solve for ψ asymptotically as $\hbar \to 0$

$$\hbar^2 \partial_x^2 \psi(x, \hbar) - q(x, \hbar)\psi(x, \hbar) = 0$$

Method:
1. solve the leading-order equation (i.e., for $\hbar = 0$)
2. add $\hbar$-dependent corrections to get the true answer for $\hbar \neq 0$

- This is a problem in ***singular perturbation theory***
- Major difficulty: step 2 is very tricky!

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1. Write down the ***WKB ansatz***: $\psi(x,\hbar) = \exp\left(- \int_{x_*}^{x} s(x',\hbar)\, \mathrm{d}x'/\hbar \right)$

2. Get a ***singularly perturbed Riccati equation*** $\hbar \partial_x s = s^2 - q$

3. Solve the Riccati equation in $\hbar$-power series: $\widehat{s}(x,\hbar) = \sum_{k=0}^{\infty} s_k(x)\hbar^k$

$$\begin{array}{l|l} \hbar^0 & s_0^2 = q_0 \\ \hbar^1 & \partial_x s_0 + 2 s_0 s_1 = q_1 \\ \hbar^2 & \partial_x s_1 + 2 s_0 s_2 + s_1^2 = q_2 \\ & \vdots \end{array}$$

No differential equations!

$\Rightarrow$ two formal solutions $\widehat{s}_\pm$ with $s_0^\pm := \pm\sqrt{q_0}$

4. Get two ***formal WKB solutions***:

$$\widehat{\psi}_\pm(x,\hbar) := \exp\left(- \int_{x_*}^{x} \widehat{s}_\pm(x',\hbar)\, \mathrm{d}x'/\hbar \right)$$

Major problem: $\widehat{s}_\pm$ **are generically divergent!**

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$$\psi_\pm^{(0)}(x,\hbar) := \exp\left(-\int_{x_*}^{x} \pm\sqrt{q_0(x')}\,\mathrm{d}x'/\hbar\right)$$

3. **Exact WKB analysis:** interpret $\widehat{s}_\pm$ as asymptotic expansions as $\hbar \to 0$ of true analytic solutions $s_\pm$, and get ***exact WKB solutions***:

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Then: $\psi_\pm \simeq \widehat{\psi}_\pm$ as $\hbar \to 0$ and $\psi_\pm^{(0)}$ is the leading-order asymptotic behaviour
OR: $s_\pm \simeq \widehat{s}_\pm$ as $\hbar \to 0$ and $s_0^\pm = \pm\sqrt{q_0}$ is the leading-order asymptotics

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where p, q are convergent $\hbar$-power series with meromorphic coefficients:

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4. **Let:** $\widehat{s}_{\pm}$ be the two formal solutions with leading-orders $s_0^{\pm}$

Theorem (Existence&Uniqueness of Exact WKB Solutions [N'2020])

Let: $\mathsf{U} \subset \mathbb{C}_x$ *simply connected domain with nonvanishing* $\Delta := p_0^2 - 4q_0$.
Assume: U *is complete under the flow of real analytic vector field* $\mathrm{Re}\left(\frac{1}{\sqrt{\Delta}}\partial_x\right)$.

Assume: p, q *are bounded by* $\sqrt{\Delta}$ *on* U

Then: *there are exactly two analytic solutions* $s_{\pm}$ *such that for all* $x \in \mathsf{U}$,

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Therefore: *fixing a basepoint* $x_* \in \mathsf{U}$,
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Then: *there are exactly two analytic solutions* $s_\pm$ *such that for all* $x \in \mathsf{U}$,

$$s_\pm(x,\hbar) \simeq \widehat{s}_\pm(x,\hbar) \quad \text{as } \hbar \to 0 \text{ uniformly in the right halfplane}$$

Therefore: *for* $x_* \in \mathsf{U}$, *there is a unique basis of exact WKB solutions* $\psi_\pm$ *on* U
Furthermore: *get integral representation as the Laplace transform*

$$s_\pm(x,\hbar) = s_0^\pm(x) + \int_0^\infty e^{-\xi/\hbar}\sigma_\pm(x,\xi)\,\mathrm{d}\xi$$

$$\sigma_\pm(x,\xi) := s_1^\pm(x) + \sum_{k=0}^\infty \int_0^\xi \phi_k^\pm(x,t)\,\mathrm{d}t \quad \text{and} \quad I_\pm[\omega](x,\xi) := \int_0^\xi \omega(x_\pm(t), \xi - t)\,\mathrm{d}t$$

$$\phi_k^\pm(x,\xi) := I_\pm\left[b_1\phi_{k-1}^\pm + \beta_1 * \phi_{k-2}^\pm + \sum_{\substack{i,j\geqslant 0\\ i+j=k-2}} b_2\phi_i^\pm * \phi_j^\pm + \sum_{\substack{i,j\geqslant 0\\ i+j=k-3}} \beta_2 * \phi_i^\pm * \phi_j^\pm\right]$$

Theorem (Existence&Uniqueness of Exact WKB Solutions [N'2020])

Let: $\mathsf{U} \subset \mathbb{C}_x$ *simply connected domain with nonvanishing* $\Delta := p_0^2 - 4q_0$.
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☺ Thank you for your attention! ☺